# Old and New Benchmarks for Relative Termination of String Rewrite Systems

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# Relative Termination (Definition, Example)

- `# SRS_Relative/Zantema_06/rel11`
  `(RULES  b p b -> b a p b , p ->= a p a , a p a a ->= p)`
- is shorthand for pair of rewrite systems
  $R = \{bpb \to bapb\}, S = \{p \to apa, apaa \to p\}$
- relation $\to_R$ *relative to* relation $\to_S$: $\quad \to_R / \to_S \quad := \quad \to_R \circ \to_S^*$
- Def: $R$ terminates relative to $S$ iff $\mathrm{SN}(\to_R / \to_S)$, Notation $\mathrm{SN}(R/S)$
  each (infinite) mixed derivation contains only finitely many $R$ steps
- ref: Jan Willem Klop 1987, Alfons Geser 1990, Hans Zantema 2004
- application: removal of rules ($D$) in modular absolute termination proofs $\mathrm{SN}(D/R) \wedge \mathrm{SN}(R \setminus D) \Rightarrow \mathrm{SN}(R)$
- application: rewriting modulo equations
- our contribution: discuss current TPDB/SRS-Relative benchmarks, discuss some methods for solution, provide new small benchmarks
- `(COMMENT [rel11] invariant after first rule:`
  `left from p more a's than right from p )`

# How to win SRS-Relative, by ignoring "-Relative"

- termcomp 22, SRS-Relative, winner: MultumNonMulta, 203 YES.
- $\mathrm{SN}(R \cup S) \Rightarrow \mathrm{SN}(R/S)$. we present: the *strictify* transformer: consider weak rules as strict, prove absolute termination: 211 YES.
- due to benchmarks ICFP-2010, Waldmann-19
- if $\mathrm{SN}(S)$, then $\mathrm{SN}(R/S) \iff \mathrm{SN}(R \cup S)$
- only in case $\neg\mathrm{SN}(S)$ do we need specific proof methods for $\mathrm{SN}(R/S)$.
- actual matchbox (2023) strategy expression (`strat/combi.strat`)

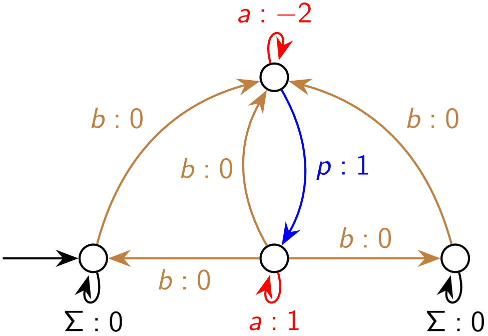
```
let { standard = ... ; relative = ... ; ... }
in Apply cleaner (Or_Else done (Apply weights  (Or_Else done
    (Or_Else
       (Apply (When_True (Apply dropstrict
                                  (Apply strictify standard)))
              (Apply strictify standard))
       relative    ))))
```

# Relative Non-Termination

- the obvious method is to find a loop $w \ (\to_R / \to_S)^+ \ pwq$
- one specific method ($R$-emitting loops, AProVE TC 22) is:
  if $S$ admits a loop $w \to_S^+ pwq$ such that $p$ or $q$ contains an $R$-Redex, then $\neg\mathrm{SN}(R/S)$.
  in this case, right-hand sides of $R$ do not matter!
- (Geser, Zantema 1999) for absolute termination:
  $R$ admits loop $\iff$ $R$ admits looping forward closure (FC)
- not true for relative termination:
  example: $\{bab \to a, c \to^= cb, d \to^= bd\}$
  has loop $cad \to^{=2} cbabd \to cad$ but no looping FC
- given loop is overlap closure (OC).
  cf. role of FC/OC in *sparse tiling* for absolute/relative termination.
  OCs are more expensive to enumerate than FCs

# Relative Termination: When in doubt—use brute force

- that is, matrix interpretation via SAT encoding (ersatz, kissat)
- arctic (below zero) matrix int. for Zantema-06/rel11 (open in TC 22)
  $\{bpb \to bapb, p \to^{=} apa, apaa \to^{=} p\}$,




$$a = \begin{pmatrix} 0 & . & . & . \\ . & -2 & . & . \\ . & . & 1 & . \\ . & . & . & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 & . & . \\ . & . & . & . \\ 0 & 0 & . & 0 \\ . & 0 & . & 0 \end{pmatrix}, p = \begin{pmatrix} 0 & . & . & . \\ . & . & 1 & . \\ . & . & . & . \\ . & . & . & 0 \end{pmatrix}$$

# Relative Termination: When in doubt—...

- Zantema-06/rel12: $\{bpb \to abapba, p \to^{=} apa, apa \to^{=} p\}$,
- natural matrix interpretation

$$a = \begin{pmatrix} 1 & . & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & 1 & . & . & . \\ . & . & . & . & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 1 & . & . & . \\ . & 1 & 4 & . & 1 \\ . & 1 & . & . & . \\ . & 2 & . & . & . \\ . & . & . & . & 1 \end{pmatrix}, p = \begin{pmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & . & 1 & . \\ . & . & 1 & . & . \\ . & . & . & . & 1 \end{pmatrix}$$

- how is the previous related to this arctic matrix interpretation:

$$a = \begin{pmatrix} 0 & . & . & . & . \\ . & . & . & 1 & . \\ . & -1 & . & . & . \\ . & . & 0 & . & . \\ . & . & . & . & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & . & 0 & . & . \\ 1 & 2 & -1 & . & 1 \\ . & . & 0 & . & . \\ . & . & -1 & . & . \\ . & . & 0 & . & 0 \end{pmatrix}, p = \begin{pmatrix} 0 & . & . & . & . \\ . & . & 2 & . & . \\ . & 0 & . & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 0 \end{pmatrix}$$

# Where brute force does not help (so far)

- (remains open) Zantema-06/rel03:

$$\{ac \to cca, c \to^{=} baab, baab \to^{=} c\}$$

  is related to $a(baab) \to (baab)^2 a$, which is RFC-matchbounded.

- (remains open) Zantema-06/cars

```
( RULES  Mr R -> Ml cr R , L Ml -> L Mr cr
, Mr o -> Ml cr , Mr n -> Ml cr , o Ml -> Mr cr , n Ml -> Mr cr
, Mr o ->= Mr , Mr n ->= Mr , o Ml ->= Ml , n Ml ->= Ml
, Ml cr ->= cl Ml , Mr cr ->= cl Mr , L ->= L n , R ->= n R
, cr n ->= n cr , cr o ->= o cr , cr o ->= o
, n cl ->= cl n , o cl ->= cl o , o cl ->= o  )
```

# Brand New: Small(est) Hard Relative SRS

- most small TPDB benchmarks are solved—then what next?
- make a complete enumeration by size, filter w.r.t. current provers
- cf. enumerations for SRS-absolute:
  - one-rule: Kurth (1990), Geser (2002), Wenzel-16,
  - many-rule: Waldmann-07
- fresh relative SRS: Waldmann-23, smallest unsolved:
  - alphabet 3: size 7

    ```
    (RULES a c -> c, ->= a b, a b ->= )
    (RULES a c -> c, ->= a b, b a ->= )
    ```

    up to size 8: 41 benchmarks, 34 unsolved
  - alphabet 2: size 9

    ```
    (RULES a a b b a -> , ->= a b a b)
    ```

    up to size 10: 57 benchmarks, 13 unsolved
- NB: starexec could run such enumerations/filterings all year long. . .

# Two New benchmarks, with Manual Proofs

- `(RULES a c -> c, ->= a b, a b ->= )`
  hand-waving: number of un-matched *a* is reduced
  exact: this number is first component of interpretation

$$\begin{aligned} a_I(x,y) &= \text{if } y > 0 \text{ then } (x, y-1) \text{ else } (x+1, 0) \\ b_I(x,y) &= (x, y+1) \\ c_I(x,y) &= (x, 0) \end{aligned}$$

  is monotone w.r.t. order $(x_1, y_1) > (x_2, y_2)$ iff $x_1 > x_2 \wedge y_1 = y_2$
- `(RULES a c -> c, ->= a b, b a ->= )`
  use the very same interpretation as above, but with order:

$$\begin{aligned} (x_1,y_1) > (x_2,y_2) &\quad \text{iff} \quad (x_1 > x_2) \wedge (y_1 \geq y_2) \wedge (x_1 - y_1 > x_2 - y_2) \\ (x_1,y_1) \geq (x_2,y_2) &\quad \text{iff} \quad (x_1 \geq x_2) \wedge (y_1 \geq y_2) \wedge (x_1 - y_1 \geq x_2 - y_2) \end{aligned}$$

- is this semantic labeling w.r.t. a (quasi) model over $\mathbb{N}$? see also Hofbauer WST'18.

# Conclusion/Discussion

- Retire/Relabel SRS-Relative/{ICFP-10,Waldmann-19}?
  - keep in TPDB but don't use in competition
  - OTOH, do use, but de-value?
- new small hard SRS:
  - solve them,
  - devise new methods to automatically solve them
- certified relative termination?
  CPF/CeTA currently has all we need, except for:
  - sparse tiling, with overlap closures (has full tiling)

  and these methods for absolute termination, needed after `strictify`:
  - RFC (approximated) matchbounds (has full matchbounds)
  - sparse tiling, with forward closures

  so ... I am starting a project *verified SRS termination in Agda*

## Questions asked after the talk

- Danger: notation in the paper is misleading: uses $\epsilon$ in two meanings:
  - in rule: $\epsilon \to^= ab$, translated into TRS rule $x \to^= a(b(x))$
  - in interpretation: $\epsilon_I = (0, 0)$, epsilon denotes the nullary symbol in the leaf of a term (tree) that encodes a string ($abc$ encoded as $a(b(c(\epsilon)))$
- Q: Do you have a theorem about "$R/S$ is looping $\iff$ $R/S$ has a looping overlap closure"?
  A: No. — We have (FSCD19)
  "$\text{SN}(R/S) \iff \text{SN}(R/S, \text{ROC}(R \cup S))$" (for relative termination, it is enough to consider mixed derivations strarting from right-hand sides of overlap closures)
- Q: Kissat over Minisat—did you measure?
  A: I guess I did but I did not take detailed notes.
- Q: Why the new solutions (rel11, rel12)?
  A: change in proof search strategy. Matchbox has too many moving, and moveable parts. Changes in strategy expression may have unforseen consequences.