# Approximating Relative Match-Bounds

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## Motivation

**The 595 problems from `TPDB/SRS_STANDARD/ICFP_2010` are**

- large: avg. 70 rules of size 2340 (non-`ICFP`: 3.3 of size 21.5)
- time consuming: VBS CPU time at termCOMP'21
  avg. 90", median 28" (non-`ICFP`: avg. 51", median 6")
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**termCOMP'21 versus '22**

| | Matchbox | MnM | VBS |
|---|---|---|---|
| termCOMP'21 | 510 | 417 | 514 |
| termCOMP'22 | 595 | 594 | 595 |



## Overview

**Methods from this talk** (timeout 10")

| | **rb** | **rel. rb** | **mb** | **rel. mb** |
|---|---|---|---|---|
| solved | 370 | 568 | 588 | 590 |
| % | 62.2 | 95.5 | 98.8 | 99.2 |
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- Idea: remove relatively (on RFC) match-bounded rules (H/W'10)
- New: approximate this property fast
- Ingredients: (Dershowitz'81); (Büchi'64); (McNaughton'94, Geser'01); automata completion (various authors)
- Independent implementations in Matchbox and MnM



## Termination of (string) rewriting

**Modular termination proofs by removing rules**

- SN($R$): $R$ is *terminating* (or: *strongly normalizing*) if every $R$-derivation contains only finitely many $R$-steps.
- SN($R/S$): $R$ is *terminating relative to* $S$ if every $(R \cup S)$-derivation contains only finitely many $R$-steps.
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**How to prove SN($R$), or prove SN($R/S$)?**

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- Ad hoc approach: $0 \in$ finitely many. Show that $R$-steps do not occur in any $R$-derivation, or show that $R$-steps do not occur in any $(R \cup S)$-derivation.
- Nonsensical, this is never the case . . . . . . but could work for a restricted set of derivations.



## Restricting the set of derivations

### Definition: Right-hand sides of forward closures

- $\mathrm{RFC}(R) = (\to_R \cup \rightharpoonup_{\mathrm{right}(R)})^*(\mathrm{rhs}(R))$, where $\rightharpoonup$ is suffix rewriting, and $\mathrm{right}(R) = \{\ell_1 \to r \mid (\ell_1\ell_2 \to r) \in R, \ell_1 \neq \epsilon \neq \ell_2\}$.
- $\to_R$ are *inner steps*, $\rightharpoonup_{\mathrm{right}(R)}$ are *suffix extension steps*.



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### Example: $R = \{ab \to ba\}$

Here, $\mathrm{right}(R) = \{a \to ba\}$, so $\mathrm{RFC}(R) = (\to_R \cup \rightharpoonup_{\mathrm{right}(R)})^*(ba) = b^+a$. RFC($R$) contains no $R$-redex, so $R$ is terminating.



## Right barren string rewriting

Generalizing McNaughton'94, Geser'01 from 1-rule to arbitrary finite systems:

### Definition: $R$ is *right barren*

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### Proof of decidability

If $R$ is right barren, $\mathrm{RFC}(R) = \rightharpoonup_{\mathrm{right}(R)}{}^*(\mathrm{rhs}(R))$. This set is regular, since regularity is preserved under suffix rewriting (Büchi'64).



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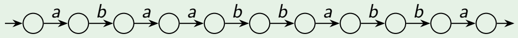
Automaton accepting rhs($R$):



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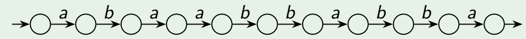
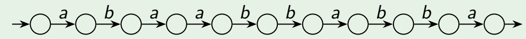



Closure under $\to_{\text{right}(R)}$ by adding epsilon transitions:



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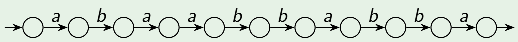


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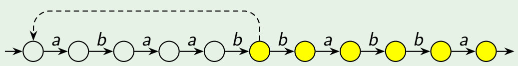




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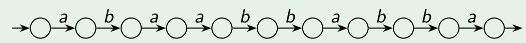


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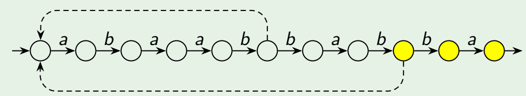




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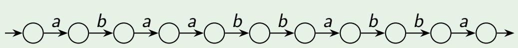


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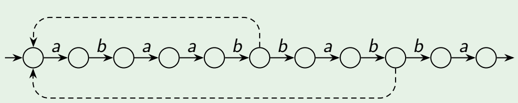


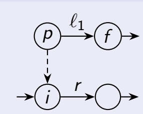

The left-hand side of $R$ is not a factor of any accepted string, so $R$ is right barren, thus terminating.



## Right barren string rewriting (cont'd)

**Closure algorithm: suffix matches**

For state $p$, final state $f$, $(\ell_1 \to r) \in \text{right}(R)$:
If there is a path $p \xrightarrow{\ell_1} f$, add $p \xrightarrow{\epsilon} i$,
where $i$ is the initial state of the path for $r$.



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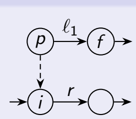


- Termination of this algorithm: No new nodes, so there are only finitely many possible epsilon transitions.
- Decide whether $\ell \in lhs(R)$ is a factor of some accepted string: check for path $p \xrightarrow{\ell} q$ (states are accessible and co-accessible).



## Removing relatively right barren rules

**Definition: $S \subseteq R$ is *relatively right barren* w. r. t. $R \setminus S$**

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**Theorem:** Let $S \subseteq R$ be relatively right barren w. r. t. $R \setminus S$.
Then $\mathrm{SN}(R \setminus S)$ implies $\mathrm{SN}(R)$.



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**Closure algorithm: suffix and redex matches**

Closure steps for suffix matches as before.
Closure steps for redex matches:
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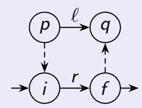




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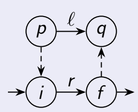


The resulting automaton over-approximates $\mathrm{RFC}(R)$.



## Removing relatively right barren rules (cont'd)

**Example:** $R = \{ab \to ba, ba \to acb\}$ (`Zantema_04/z006`)

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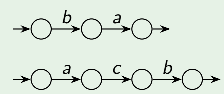




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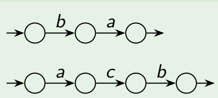


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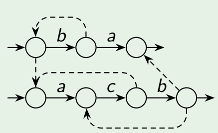




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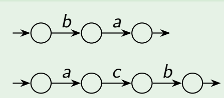


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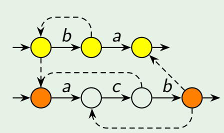




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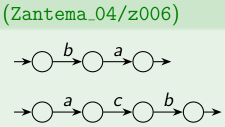


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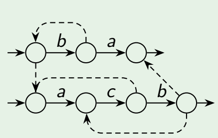


There is no path labelled by the left-hand side of $S = \{ab \to ba\}$:
$S$ is relatively right barren w. r. t. $R \setminus S$. As $R \setminus S = \{ba \to acb\}$ is terminating (it is right barren), $R$ is terminating.



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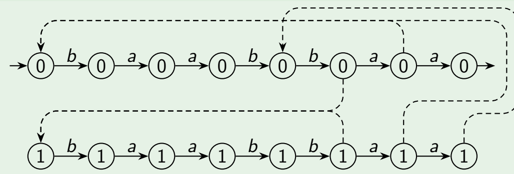
- In case of success: complete automaton is a certificate for match-bound $B$ on RFC($R$).



## Approximating match-bounds (cont'd)

**Example:** $R = \{abaab \to baabbaa\}$ (`Zantema_04/z034` reversed)



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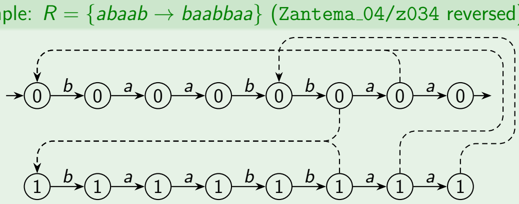

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**Removing relatively match-bounded rules (sketch)**

- Now layer $B$ represents all heights $\geq B$; we never reject.
- After completion, remove those rules where all redex heights are $< B$: they are match-bounded relative to the remaining rules by $B$ on RFC, so they terminating relative to the remaining rules.



## Summary and discussion

- This method solves `SRS_STANDARD/ICFP_2010`. Weaker on non-`ICFP`: Solves 164 of 1056.
- Cannot solve `Zantema_04/z001`.
- But, by iteration, solves problems that are not (RFC-)match-bounded.
- Two independent implementations: Confidence, no certification.
- Combined with *drop common prefix/suffix*, nearly solves `Wenzel_16`: MnM solves 222 of 226.
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- Challenge: merge this method with the exact RFC-method (Endrullis/H/W'06).
- Challenge: termCOMP needs more SRS benchmarks — that are independent of any specific method. Continue systematic or random enumeration.