Sparse Tiling Through Overlap Closures for Termination of String Rewriting

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Preliminaries: Termination

- relation $\to$ is terminating (strongly normalizing)
  := there are no infinite $\to$-chains
  notations: $\mathrm{SN}(\to), \mathrm{SN}(\to_R), \mathrm{SN}(R)$.
- methods for proving termination of rewriting:
  - syntactical (precedence on symbols)
  - semantical (interprete symbols by functions over well-founded domain)
  - transformational ($\mathrm{SN}(R) \Leftarrow \mathrm{SN}(R')$)
- in particular: transformation that increases signature, give more room to pick predecence or interpretation
- …by *tiling*: new signature consists of *tiles* (blocks of adjacent letters)

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- $S = \{aa \to aba\}$ does not remove letters
- use tiles of width 2 (pairs of adjacent letters)
  $S_2 = \{[aa] \to [ab, ba]\}$, can simulate S-derivations
  S_2 removes letter $[aa]$: is terminating!
- in general: need (left and right) padding
  ex. from rule $ab \to ba$, create
  $[aa][ab][ba] \to [ab][ba][aa]$, $[aa][ab][bb] \to [ab][ba][ab]$,
  $[ba][ab][ba] \to [bb][ba][aa]$, $[ba][ab][bb] \to [bb][ba][ab]$
- instance: root labelling (Sternagel, Middeldorp, RTA 2008)
- our contribution:
  - use smaller set of tiles (for rewriting and for padding)
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Sparse Tiling: Definition and Motivation

- Ex. the bordered 3-tiles of string $w = bbaab$ are $\mathrm{btiles}_3(w) = \{\triangleleft\triangleleft b, \triangleleft bb, bba, aab, ab\triangleright, b\triangleright\triangleright\}$
- Def. [Zalcstein 1972] strictly locally testable language $\mathrm{Lang}(T) = \{w \mid \mathrm{btiles}(w) \subseteq T\}$
- this paper:
  - use such languages to over-approximate $R^*(L)$
  - represent T by finite automaton A,
  - … constructed by completion
  - semantically label R by the partial algebra of A
  - … to transform the termination problem of R on L
  - *sparse*: T is the set of tiles that occur in rhs of forward closures (overlap closures, resp.)
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Right-hand Sides of Forward Closures

- Def. $\mathrm{RFC}(R) =$ smallest set $M \subseteq \Sigma^*$ with
  - (start) $\mathrm{rhs}(R) \subseteq M$
  - (inner step) $(l, r) \in R \wedge ulv \in M \Rightarrow urv \in M$
  - (right extension) $(l_1 l_2, r) \in R \wedge ul_1 \in M \Rightarrow ur \in M$
- Thm. (Dershowitz 1981)
  R terminates on $\Sigma^* \iff R$ terminates on $\mathrm{RFC}(R)$
- Ex. $\mathrm{RFC}(\{ab \to ba\}) = b^+ a$. Cor.: is terminating.
- Lemma: $\mathrm{RFC}(R) = (R \cup \mathrm{forw}(R))^*(\mathrm{rhs}(R))$ where
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Representing Sets of Tiles by Automata

- Def: the k-shift automaton
  (it remembers $k-1$ most recent letters read)
  alphabet $\Sigma \cup \{\triangleright\}$,
  states $\text{tiles}_{k-1}(\triangleleft^*\Sigma^*\triangleright^*)$, initial state $\triangleleft^{k-1}$, final state $\triangleright^{k-1}$,
  transitions: $p \xrightarrow{c}_A \text{Suffix}_{k-1}(pc)$
- represents set of k-tiles $\text{tiles}(A) := \{pc \mid p \xrightarrow{c}_A q\}$

- Ex. 2-shift automaton $A =$
  represents 2-tiles $\{\triangleleft a, \triangleleft b, ab, ac, bb, bc, c\triangleright\}$
  $\text{Lang}(A) = (a+b)b^*c$

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  represents 2-tiles $\{\triangleleft a, \triangleleft b, ab, ac, bb, bc, c\triangleright\}$
  $\text{Lang}(A) = (a+b)b^*c$

Representing Sets of Tiles by Automata

- Def: the k-shift automaton
  (it remembers $k-1$ most recent letters read)
  alphabet $\Sigma \cup \{\triangleright\}$,
  states $\text{tiles}_{k-1}(\triangleleft^*\Sigma^*\triangleright^*)$, initial state $\triangleleft^{k-1}$, final state $\triangleright^{k-1}$,
  transitions: $p \xrightarrow{c}_A \text{Suffix}_{k-1}(pc)$
- represents set of k-tiles $\text{tiles}(A) := \{pc \mid p \xrightarrow{c}_A q\}$
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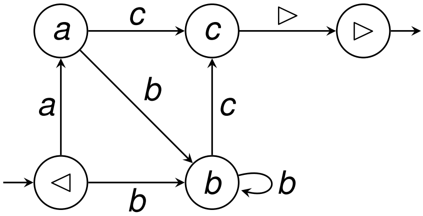

  represents 2-tiles $\{\triangleleft a, \triangleleft b, ab, ac, bb, bc, c\triangleright\}$
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Rewrite Closure of Tiling Automata

- spec: given k-shift A, R over Σ, find k-shift A' over Σ s.t.
  - $\mathrm{Lang}(A) \subseteq \mathrm{Lang}(A')$
  - $u \in \mathrm{Lang}(A') \land u \to_R v \Rightarrow v \in \mathrm{Lang}(A')$
- implementation:
  when $(l, r) \in \mathrm{CC}_k(R)$ (right k-context closure)
  and $p \xrightarrow{l}_A q$,
  add transitions and states such that $p \xrightarrow{r}_A q$,
  until it stabilises
- by the k-shift property:
  - given p and r, the path $p \xrightarrow{r}_A q$ is fully determined,
    and it will indeed end in q
  - completion terminates since set of states is finite

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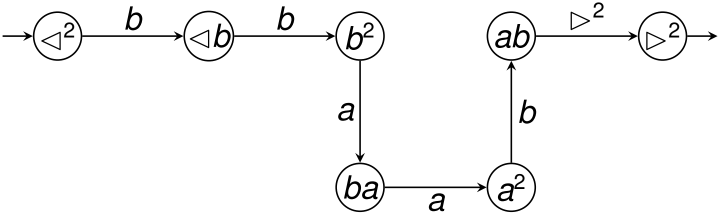
Closure Example

- for $R = \{ab^3 \to bbaab\}$,
  compute 3-shift approx. of $(R \cup \text{forw}(R))^*(\text{rhs}(R))$



- …this is the path for $\text{rhs}(R)$
- absent: $\triangleleft^2\triangleright, \triangleleft\triangleright^2, \triangleleft\Sigma\triangleright$, $\triangleleft a\Sigma, \triangleleft ba, \Sigma a\triangleright$, a^3

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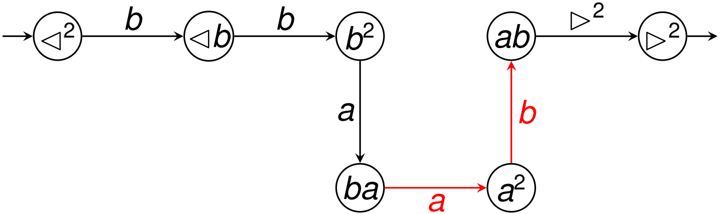

- $\to$ a redex for $(ab \to_{\text{Suffix}} bbaab) \in \text{forw}(R)$
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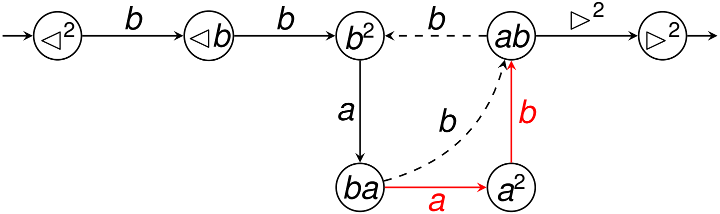

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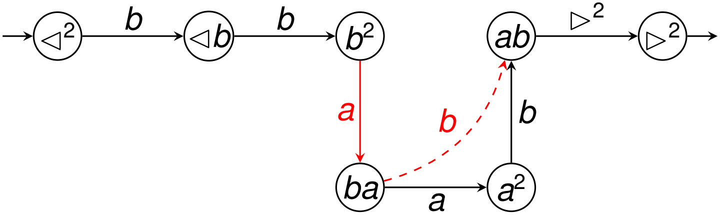

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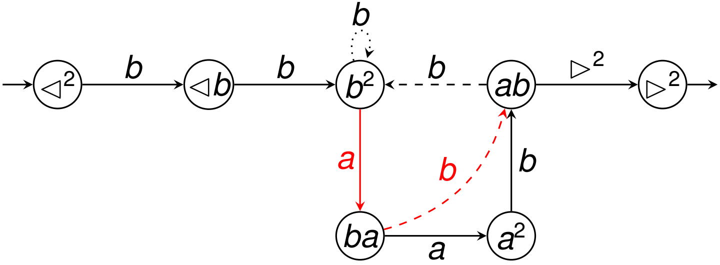

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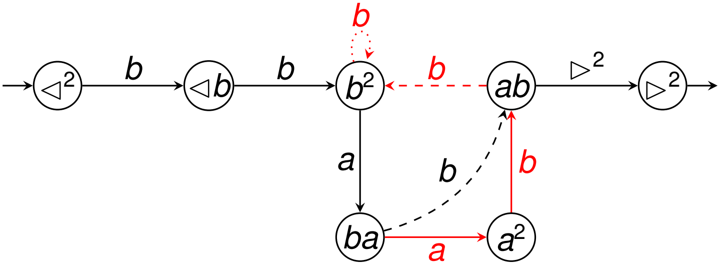

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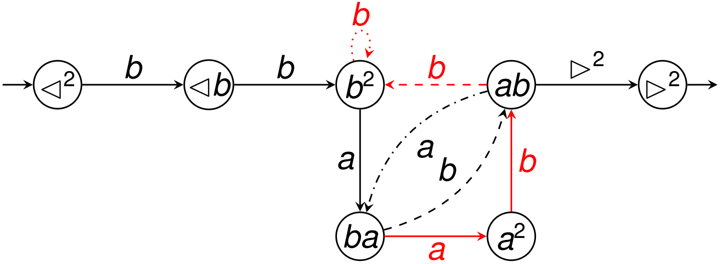

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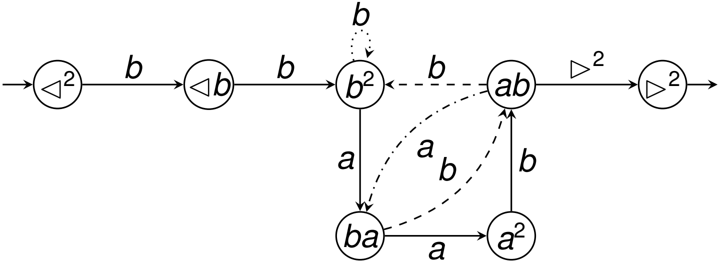

- represents the set of tiles $T = \{\lhd\lhd b, \lhd bb, bba, bbb, baa, bab, aab, aba, abb, ab\rhd, b\rhd\rhd\}$.
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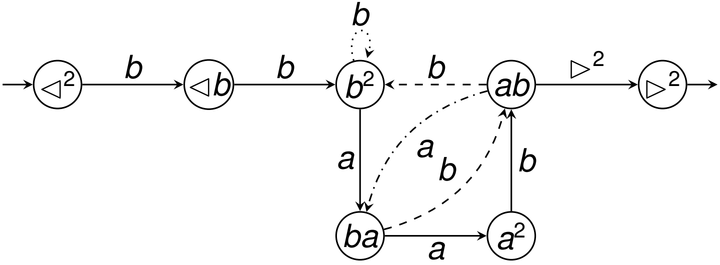

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Semantic Labelling

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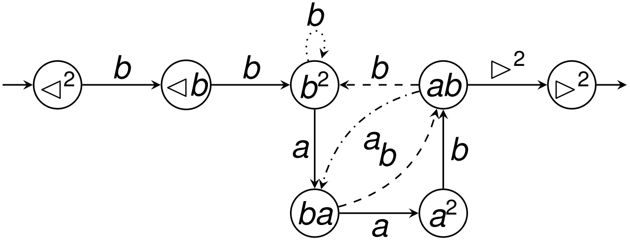

- semantically labelled R is $R' =$

$$bba, bab, abb, b^3, bbx, bxy \to b^3, b^3, bba, baa, aab, abx, bxy$$
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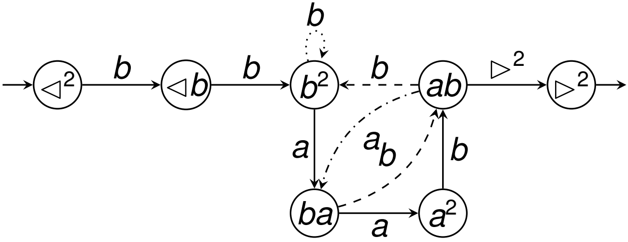

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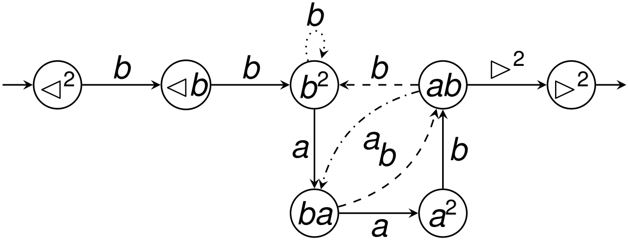

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Removing unreachable rules (Prop. 5.3)

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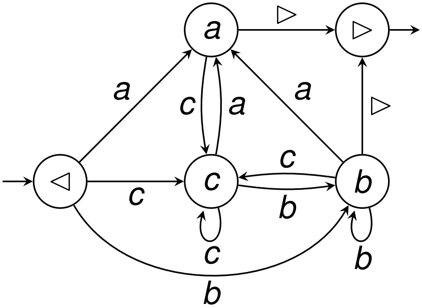

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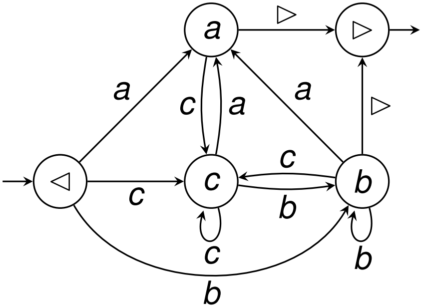

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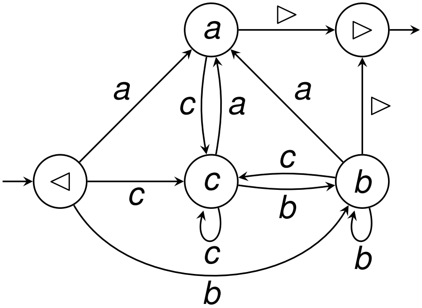

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- Theorem: each paper on SRS termination contains a termination proof for Zantema's ($\approx$ 1993) problem
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Overlap Closures and Relative Termination

- Def: R terminates relative to S, notation: $\mathrm{SN}(R/S)$, if there is no $(R \cup S)$-derivation with infinitely many R steps.
  Ex: $\mathrm{SN}(aa \to aba / a \to aba)$.
- (recap) $\mathrm{SN}(R)$ iff $\mathrm{SN}(R)$ on $\mathrm{RFC}(R)$.
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  $R = \{ab \to a\}, S = \{c \to bc\}, \mathrm{RFC}(R \cup S) = a \cup b^+c$.
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- Thm 6.7 $\mathrm{SN}(R/S)$ iff $\mathrm{SN}(R/S)$ on $\mathrm{ROC}(R \cup S)$.
  using right-hand sides of *overlap* closures
- apply left-recursive characterisation of ROC (overlap closure with rule) (see Appendix of paper).
- interesting case: (Cor 7.1.5)
  if $tx \in S$ and $yv \in S$ and $(xwy, z) \in R$, then $tzv \in S$

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Overlap Closures and Relative Termination

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Example: Tiling for Overlap Closures

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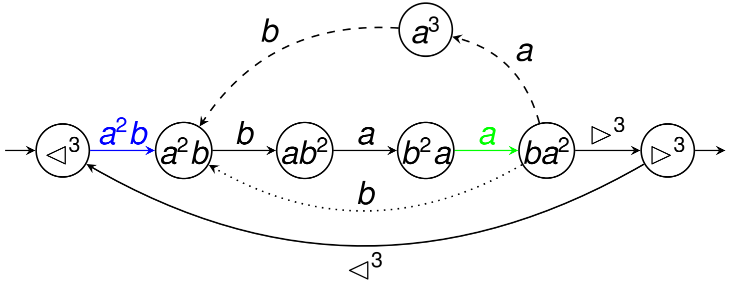

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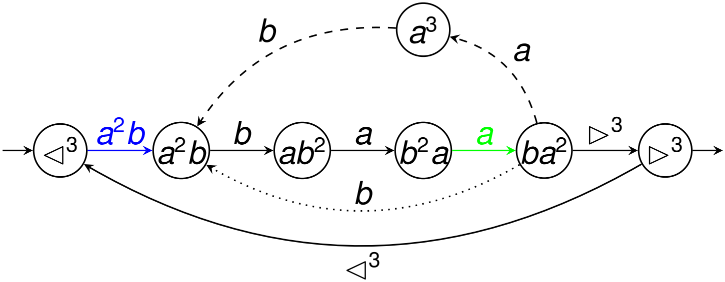

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  `https://gitlab.imn.htwk-leipzig.de/waldmann/pure-matchbox`
- performance, including *Termcomp 2019* (SRS)

Relative		matrices no	matrices yes
tiling	no	1	72
tiling	yes	176	*225*

Standard		MB, DP, matr. none	MB, DP, matr. all
tiling	no	100	1122
tiling	yes	512	*1133*

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