One-Dimensional Tiling Systems and Termination of String Rewriting

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Theorietag 2018

Termination of String Rewriting

- *string rewriting system* (SRS) R: set of rules
  examples: $R = \{ab \to bc\}, S = \{aa \to aba\}$
- *rewrite relation* $\to_R$: apply rule in context
  $aaba \to_R abca \to_R bcca$,
  $aaaa \to_S aabaa \to_S ababaa \to_S abababa$.
- relation $\to$ is *terminating* iff there are no infinite $\to$-chains
- relation $\to$ is *terminating on M* iff there are no infinite $\to$-chains starting in M
- $\to_R$ is terminating: count occurences of a
  $\to_S$ is terminating? does not remove letters!

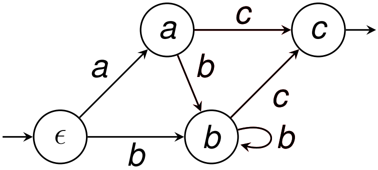
Tiling

- $S = \{aa \to aba\}$ does not remove letters
- use tiles of width 2 (pairs of adjacent letters)
  $S_2 = \{[aa] \to [ab, ba]\}$, can simulate S-derivations
  S_2 removes letter $[aa]$: is terminating!
- in general: need (left and right) padding
  ex. from rule $ab \to ba$, create
  $[aa][ab][ba] \to [ab][ba][aa], [aa][ab][bb] \to [ab][ba][ab],$
  $[ba][ab][ba] \to [bb][ba][aa], [ba][ab][bb] \to [bb][ba][ab]$
- this talk (new): use smaller set of tiles for padding:
  only those that appear in (certain) infinite derivations

Right-hand Sides of Forward Closures

- Def. $\text{RFC}(R) =$ smallest set $M \subseteq \Sigma^*$ with
  - (start) $\text{rhs}(R) \subseteq M$
  - (inner step) $(l, r) \in R \wedge ulv \in M \Rightarrow urv \in M$
  - (right extension) $(l_1 l_2, r) \in R \wedge ul_1 \in M \Rightarrow ur \in M$
- Thm. (Dershowitz 1981)
  R terminates on $\Sigma^* \iff R$ terminates on $\text{RFC}(R)$
- Ex. $\text{RFC}(\{ab \to ba\}) = b^+a$. — Cor.: is terminating.
- Plan for rest of this talk:
  - over-approximate $\text{RFC}(R)$ by a tiling system T
    - represent set of tiles by finite automaton A
    - add transitions and states according to Def. RFC
    - completion terminates by choice of set of states
  - R-derivations from $\text{RFC}(R) \sim \text{Tiled}_T(R)$-derivations

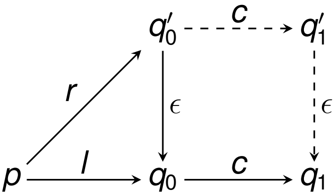
Representing Sets of Tiles by Automata

- bordered tiles, Ex. $\text{Tiles}_2(\triangleleft abbbc\triangleright) = \{\triangleleft a, ab, bb, bc, c\triangleright\}$
- for set T of bordered tiles of width k over Σ
  Def. $\text{Lang}(T) := \{w \mid \text{Tiles}_k(\triangleleft w\triangleright) \subseteq T\}$
  strictly locally testable language (Zalcstein 1972)
- automaton with states $Q(A) \subseteq \Sigma^{<k}$, initial state ϵ,
  transitions: $p \xrightarrow{c}_A$ suffix of length $\min(k-1, |pc|)$ of pc
- represents set of tiles $\text{Tiles}(A) = \{pc \mid p \xrightarrow{c}_A q\}$
  bordered tiles from initial state, final states
- Prop. $\text{Lang}(A) = \text{Lang}(\text{Tiles}(A))$
- 

  $\text{Tiles}(A) =$
  $\{\triangleleft a, \triangleleft b, ab, ac, bb, bc, c\triangleright\}$
  $\text{Lang}(A) = (a+b)b^*c$

Rewrite Closure of Tiling Automata

- specification: given A, R over Σ, find A' over Σ such that
  - $\mathrm{Lang}(A) \subseteq \mathrm{Lang}(A')$
  - $u \in \mathrm{Lang}(A') \wedge u \to_R v \Rightarrow v \in \mathrm{Lang}(A')$
- approach: $A' = \lim A_i$ where $A_0 = A$,
  - when $(l, r) \in R$ and $p \xrightarrow{l}_{A_i} q$ and $\neg(p \xrightarrow{r}_{A_i} q)$, add transitions and states such that $p \xrightarrow{r}_{A_{i+1}} q$.
- cannot apply directly because of determinism.

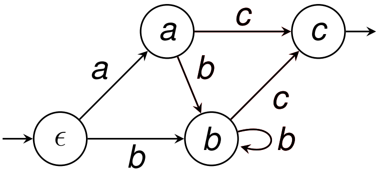


Introduce ϵ transitions, push them to the right, until $q'_k = q_k$

Closure Example and Application

- $R = \{cc \to bc, ba \to ac\}$, right ext. $\{c\triangleright \to bc, b\triangleright \to ac\}$

  RFC(R) $\subseteq$ Lang(A) for



- $\text{Tiled}_k(l, r) = \{(\text{Tiled}_k(xly), \text{Tiled}_k(xry)) \mid x, y \in \triangleleft^? \Sigma^{<k} \triangleright^?\}$
  derivations w.r.t. R and $\text{Tiled}_k(R)$ are bi-similar
- $\text{Tiled}_T(l, r) = \text{Tiled}_k(l, r) \cap T^* \times T^*$,
- Thm: if Lang(T) is closed w.r.t. R,
  then R terminates on Lang(T) iff $\text{Tiled}_T(R)$ terminates.
- in Ex., $\text{Tiled}_T(R) = \emptyset$, R terminates on RFC(R), thus, on Σ^*.

Implementation, Extension

- implemented as part of termination prover `https://gitlab.imn.htwk-leipzig.de/waldmann/pure-matchbox`
- solves classical test case $\{a^2b^2 \to b^3a^3\}$ quickly (e.g., using 336 tiles of width 12)
- can solve some (randomly generated) benchmarks that other termination provers can't . . . and vice versa
- extension: use *overlap closures* for *relative termination* solves `rbeans`: $\{baa \to abc, ca \to ac, cb \to ba\}/\{\epsilon \to b\}$ (open in all previous Termination Competitions)
- ongoing work: relation of our "tiling and closures" method to lengths of derivations