Matrix Interpretations on Polyhedral Domains

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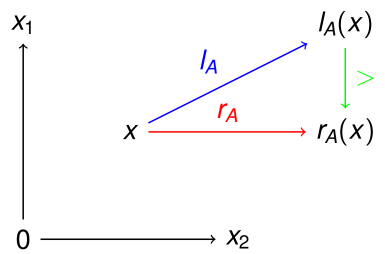
Matrix Interpretations

- domain $\mathbb{N}^d$, wellfounded $(>) := (>) \times (\geq)^{d-1}$
- linear Σ-algebra A on $(\mathbb{N}^d, >)$:
  - $\epsilon_A \in \mathbb{N}^d$,
  - for each $f \in \Sigma$ have $f_A : x \mapsto F_0 + F_1 \cdot x$
- if for each f, topleft$(F_1) > 0$, then A is monotone: $x > y$ implies $f_A(x) > f_A(y)$
- A is compatible with rule $l \to r$: $\forall x \in \mathbb{N}^d : l_A(x) > r_A(x)$
- if set of rules R admits a compatible monotone linear algebra, then R is terminating.

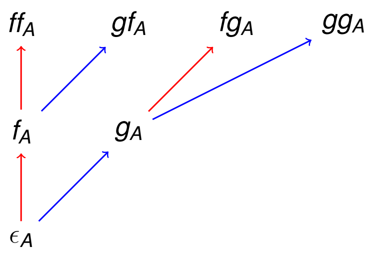
We want to improve on "$\forall x \in \mathbb{N}^d$" in compatibility

Visualizing Compatibility

- A is compatible with $l \to r$: $\forall x \in \mathbb{N}^d : l_A(x) > r_A(x)$
- 

- plan: require this only of $x \in A(\Sigma^*)$

Visualizing Reachability



- $T = A(\Sigma^*)$
- all "redex triangles" sit on points in this tree
- find polyhedral domain $D \supseteq T$ verify $\forall x \in D : l_A(x) > r_A(x)$

Polyhedral Domains

- standard method uses domain $(\mathbb{N}^d, >)$, now restrict to some subset $D \subset \mathbb{N}^d$ defined by a conjunction of linear inequalities
- D contains the weight vectors reachable by A behaviour of transitions of A outside D is ignored
- relaxed proof obligations for compatibility $\forall x \in D : [l](x) > [r](x)$ additional proof obligations $D \neq \emptyset, \forall a \in \Sigma : [a](D) \subseteq D$
- get *more* and *different* termination proofs
- idea appeared in: Lucas and Meseguer AISC'14 new: certification, implementation, extensions

Polyhedral Constraints, Example

Prove termination of $R = \{fg \to ff, gf \to gg\}$.
Use domain $D = \{(x_1, x_2, x_3) \in \mathbb{N}^3 \mid x_3 \geq x_2 + 1\}$.

$$[f](x_1, x_2, x_3) = (x_1 + 2x_2 + 1, 0, x_3 + 1)$$
$$[g](x_1, x_2, x_3) = (x_1 \quad , x_3, x_3 + 1)$$

$$[fg](x) = (x_1 + \quad 2x_3 + \boxed{1}, 0, x_3 + 2),$$
$$[ff](x) = (x_1 + \boxed{2}x_2 + \quad \boxed{2}, 0, x_3 + 2).$$

Now $\forall x \in D : [fg](x) > [ff](x)$, despite $\boxed{2}$.
$x_1 + 2x_3 + 1 \geq x_1 + (2x_2 + \boxed{2}) + \boxed{1} > x_1 + 2x_2 + 2$

Interpret. with Polyhedral Constraints

A polyhedrally constrained matrix interpret. contains:

- the interpretation, $f_A(x_1, \dots) = F_0 + \sum F_i x_i$
- the domain, given by $C_A \in \mathbb{Q}^{c \times d}, B_A \in \mathbb{Q}^{c \times 1}$, as $D = \{x \mid x \geq 0, Cx + B \geq 0\} \subseteq \mathbb{N}^d$

In the example, $d = 3, c = 1, C = (0, -1, 1), B = -1$.

to use it for termination of rewriting, we show:

- domain is non-empty,
- interpretation respects the domain,
- interpretation is compatible with rules.

for each of these, we use *certificates*

Polyhedral Constraints: Domains

Def: *A respects the domain* if $f_A : D^k \to D$.
This is certified by giving

- for each letter f, with interpretation $f_A(x_1, \dots) = F_0 + \sum F_i x_i$,
- matrices $W_1, \dots W_k \in \mathbb{Q}_{\geq 0}^{c \times c}$ with $CF_0 + B \geq (\sum_i W_i)B$, $\forall 1 \leq i \leq k : CF_i \geq W_i C$

example: $D = \{(x_1, x_2, x_3) \in \mathbb{N}^3 \mid x_3 \geq x_2 + 1\}$,
$[f](x_1, x_2, x_3) = (x_1 + 2x_2 + 1, 0, x_3 + 1)$
take $W_1 = 0$

Polyhedral Constraints: Compatibility

Compatibility of A w.r.t. rule $(l \to r)$
with $|\mathrm{Var}(l) \cup \mathrm{Var}(r)| = k$
where $([l]_A - [r]_A)(x_1, \ldots, x_k) = \Delta_0 + \sum_i \Delta_i x_i$,

is certified by matrices $U_1, \ldots, U_k \in \mathbb{Q}_+^{d \times c}$,
such that $\forall i : \Delta_i \geq U_i C$ and $\Delta_0 > \sum_i U_i B$

example: $D = \{\vec{x} \in \mathbb{N}^3 \mid -x_2 + x_3 - 1 \geq 0\}$,
$[f](\vec{x}) = (x_1 + 2x_2 + 1, 0, x_3 + 1)$,
$[g](\vec{x}) = (x_1, x_3, x_3 + 1)$,
$[fg - ff](\vec{x}) = (-2x_2 + 2x_3 - 1, 0, 0)$
take $U_1 = (2, 0, 0)^T$.

Polyhedral Constraints: Combined

to prove termination of rewriting system R,
determine

- matrix interpretation (weighted automaton)
- polyhedral domain (linear inequalities)

as solution of a constraint system for validity of certificates for

- non-emptiness of the domain
- respecting the domain
- compatibility with rules

implemented in termination prover Matchbox2015.

Completeness of Certificates

Thm: automaton respects domain, is R-compatible
$\iff$ certificates exist.

- Correctness ("$\Leftarrow$") is easily verified.
- Completeness ("$\Rightarrow$") follows from (inhomogenous) Farkas' Lemma.

The Lemma (in one of many versions) says

- A linear inequalitiy I is implied by a system S of linear inequalities
- $\iff$ $I \geq$ some positive linear combination of S.

Derivational Complexity

- by restricting the set of matrices allowed in interpretations (e.g., upper triangular),
  one restricts the growth of matrix products (e.g., to polynomial)
  and obtains bounds on derivational complexity
- polyhedral domain restriction is orthogonal to this idea, combination is sometimes helpful
- ex. $R = \{fg \to ff, gf \to gg\}$: given automaton is upper triangular, this proves $\mathrm{dc}(R)$ quadratic, this was known, but by different method (root labelling)

Dependency Pairs and Polyhedral D.

...can be easily combined. — For Usable Rules:

- need C_E-termination: add fresh symbol C, interpretation should be compatible with $C(x, y) \to x, C(x, y) \to y$,
- domain D must verify: $x, y \in D \Rightarrow \sup(x, y) \in D$, this is not always the case, e.g.,
  $D = \{(x_1, x_2) \mid 0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 2\}$
  $\sup((2, 0), (0, 2)) = (2, 2) \notin D$
- sufficient criterion: at most one coeff. < 0
- could use something better here

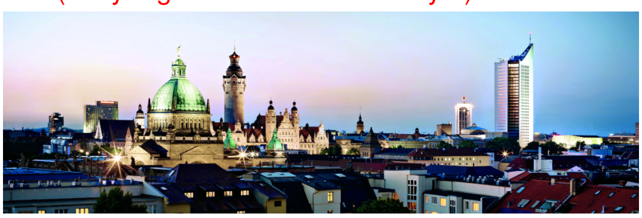
Results, Discussion

- method is correct, implementation works
- found some termination and complexity proofs where no plain matrix proof is known.
- challenge: improve implementation (improve constraint solver, better bit-blasting)
- challenge: could this method prove quadratic derivational complexity of z086?
  $\{a^2 \to bc, b^2 \to ac, c^2 \to ab\}$
- open: extend method to other (exotic) semirings, using results from tropical geometry.
- announcements: ISR 2015, termCOMP

International School on Rewriting

`http://nfa.imn.htwk-leipzig.de/ISR2015/`

- ISR 2015 at HTWK Leipzig, August 10-14.
- basic track: full introductory course,
  advanced track: 8 short courses
- you can still register your students — do it NOW! (early registration deadline: July 1)



Termination Competition 2015

`http://termination-portal.org/`

- registration of solvers: July 1
- submission of new TPDB problems: July 7
- updates of solvers: July 15
- competition runs: August 5/6 (during CADE)
- informal meeting for competitors: tonight