Termination and Complexity
Lesson 2
Intl. School on Rewriting 2014

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Day 2

- polynomially growing matrix interpretations
  Waldmann: RTA 10, `http://dx.doi.org/10.4230/LIPIcs.RTA.2010.357`
- relative termination and complexity
  Zankl/Korp: RTA 10, LMCS 10(1:19)2014
  `http://cl-informatik.uibk.ac.at/users/hzankl/new/publications/ZA14_01.pdf`

Multilinear Algebras and Paths

Recall multilinear algebra with functions of shape
$f(x_1^T, \ldots) = v + \sum M_i \cdot x_i^T$
by distributivity, $t_A = \sum\{\text{Path}_A(t, p) \mid p \in \text{Pos}(t)\}$
where $\text{Path}_A(t, p) =$ the product of the matrices in A along the path from root to p, finally multiplied by the vector (absolute coefficient) at p.

Multilinear Algebras and Bounds

Corollary: t_A bounded (component-wise) by $|t| \cdot \max_p \mathrm{Path}_A(t, p)$ bounded by $M_1 \cdot \ldots \cdot M_k$ where $M_i \in$ the matrices of the interpretation and $k =$ the depth of the term ($\leq$ than the size).

Consequently, this problem is essential for bounding derivational complexity of rewrite systems that admit a matrix interpretation:

- given a set of matrices $M \subseteq \mathbb{N}^{d \times d}$,
- define $\mathrm{growth}_M(k) = \max\{\max P \mid P \in M^k\}$
- question: is growth_M bounded by a polynomial?
- …and if yes, then give its degree.

Growth of matrix monoids (Examples)

- $\{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\}$ is linear
- $\{\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}\}$ is linear
- $\{\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\}$ is quadratic
- $\{\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}\}$ is ?

Matrix monoids and Singletons

Thm. growth of set $\{M_1, \ldots, M_k\}$ is bounded by growth of singleton set $\{\max_i M_i\}$ (point-wise max).

Thm. growth of singleton $\{M\}$ is polynomial iff eigenvalues of M are in or on unit circle.

Example: eigenvalues and -vectors of $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Matrix monoids and Singletons

Note: this reduction to singletons may destroy polynomiality, compare growth of
$\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\}$ and $\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\}$.
for the following matrix shape, this does not happen

Upper Triangular Matrix Monoids

Def: $M \in \mathbb{N}^{d \times d]}$ is upper triangular if

- below main diagonal: only 0
- on main diagonal: only 0 or 1

Thm: any set of upper triangular matrices is polynomially growth-bounded, the degree is at most $d - 1$.
Note: this bound is not sharp, examples?

Deciding polynomial growth (I)

Algorithm: from $M = \{M_1, \ldots, M_k\} \subseteq \mathbb{N}^{d \times d}$, construct graph G (automaton) with states $\{1, \ldots, d\}$ and alphabet $\{1, \ldots, k\}$ and edges (transitions) $p \xrightarrow{i} q$ iff $M_i(p, q) > 0$.
Call an edge *red* if $M_i(p, q) > 1$ (others, *black*)
Def: a *diamond* is pair of distinct paths with identical start, label, end
Thm. if an SCC of G contains a red edge, or a (black) diamond, then growth is exponential — else, polynomial.

Deciding polynomial growth (II)

checking for diamonds
use $G \times G$ (the cartesian product automaton)

Deciding polynomial growth (Exerc.)

- Thm. set of matrices $M \subseteq \mathbb{N}^{d \times d}$ grows polynomially $\iff$ in each product of M matrices, the main diagonal elements are $\in \{0, 1\}$
- What is the (best) degree of the polynomial bound, obtained from this method? We can take "height of DAG of SCCs" but it need not be sharp.

Lex. Comb. and Deriv. Complexity

We have $\mathrm{SN}(R/S) \wedge \mathrm{SN}(S) \Rightarrow \mathrm{SN}(R \cup S)$.
(we can obtain this by a lex. comb. of orders $>_R$ and $>_S$ such that $\to_R \subseteq >_R$ and $\to_S \subseteq >_R^{0,1}$ and $\to_S \subseteq >_S$)
what can we infer from $\mathrm{dc}_{R/S}$ and dc_S about $\mathrm{dc}_{R \cup S}$?
in general, not much: each R step can (drastically) increase the $>_S$-height
(Exercise: by how much exactly? Examples?)
the goal is to bound this increase.

Lex. Comb. both ways

A nice observation is
$\mathrm{dc}_{R\cup S}(n) \le \mathrm{dc}_{R/S} + \mathrm{dc}_{S/R}$
actually that's an easy observation, the nice thing is that it helps.
EXAMPLES

Weight Gap Principle (warm-up)

Assume $\text{SN}(R/S) \wedge \text{SN}(S)$
and R is size-non-increasing.
$\text{dc}_{R \cup S}(n) \leq \text{dc}_{R/S}(n) \cdot \text{dc}_S(n)$
Proof? Example that bound can be reached?
Now replace “size” by some other interpretation,
and generalize (slightly).

Weight Gap Principle (for real)

Let $\mathrm{SN}(R/S) \wedge \mathrm{SN}(S)$, and $i_S : T \to \mathbb{N}$ with

- i_S is strictly decreasing for $\to_S$
- i_S has bounded increase for $\to_R$:
  there is Δ (the *weight gap*) such that
  $x \to_R y$ implies $i_S(x) + \Delta \geq i_S(y)$

Prop. Then $\mathrm{dc}_{R \cup S}(n) \leq \mathrm{dc}_{R/S}(n) \cdot \Delta + i_S(n)$

What interpretations admit a weight gap?

- example: interpretation is sum of weights of symbols

for “sum of weights”, dc_S is linear.
there are S with linear dc that admit no weight gap:
Example: $R = \{cL \to R\}, S = \{Ra \to bbR, R \to L, bL \to LA\}$. Verify that

- $\mathrm{dc}_{R/S}$ is linear (easy)
- dc_S is linear (not that easy)
- $\mathrm{dc}_{R \cup S}$ is exponential (easy)

Prepare for Exotic Semirings

We had matrices over $\mathbb{N}$, we will see matrices over other semirings.
Homework is to recall some basic concepts

- definition of semiring
- examples: natural, tropical, arctic, fuzzy
- why do form matrices a semiring again
- well-founded order and monotonicity of operations