# Termination and Complexity
*Lesson 3*
Intl. School on Rewriting 2014

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## Day 3

- exotic semirings and their matrix semirings
- arctic and tropical matrix int. for SRS
  Gebhardt, Waldmann: Act. Cyb. 19(2), http://www.inf.u-szeged.hu/actacybernetica/edb/vol19n2/Gebhardt_2009_ActaCybernetica.xml
- arctic and tropical matrix int. with DP method
  Koprowski, Waldmann, RTA 08, Act. Cyb. 19(2)
  http://www.inf.u-szeged.hu/actacybernetica/edb/vol19n2/Koprowski_2009_ActaCybernetica.xml
- matchbounded and deleting SRS, decomposition
  Endrullis, Hofbauer, Waldmann, WST 06
  http://www.imn.htwk-leipzig.de/~waldmann/talk/06/wst/decompose/

## Exotic Multilinear Functions

Let $E$ be any semiring. Consider multlti-linear functions $f_A : (E^d) \to E^d$ of shape (as before)
$f_a(x_1^T, \ldots, x_k^T) = M_1 x_1^T + \ldots + M_k x_k^T + v$
For $E =$ arctic, tropical, fuzzy semiring:

- addition (max, min) is not monotone (examples?)
- multilinear functions are not monotone (except for very special cases)

## Monotone Exotic Multilinear Functions

$f_a(x_1^T, \ldots, x_k^T) = M_1 x_1^T + \ldots + M_k x_k^T + v$
need to remove (or treat specially) all additions:

- additions of vectors (the "+" in the above line)
- additions inside dot products in matrix-vector products

## Monotone Exotic Matrix Multiplication

For $E =$ artic, tropical:
Define $>_0$ on $E$ by
$x >_0 y \iff x > y \vee x = 0_E = y$
Define $>_0$ on $E^d$ by
$(x_1 > y_1) \wedge (x_2 >_0 y_2) \wedge \cdots \wedge (x_d >_0 y_d)$
Question: find the set of matrices $M \subseteq E^{d \times d}$ such that
Prop. Multiplication by $A \in M$ is monotone w.r.t. $>$
Exercise: and show that this does not work for $(x_1 > y_1) \wedge (x_2 \geq y_2) \wedge \cdots \wedge (x_d \geq y_d)$

## Monotone Exotic Multilinear Functions

$f_a(x_1^T, \ldots, x_k^T) = M_1 x_1^T + \ldots + M_k x_k^T + v$
Prop. If $f_A$ verifies

- $k = 1$ (unary symbols) and $v = (0_E, \ldots, 0_E)^T$ and top-left $(M_1) \neq 0_E$
- or $k = 0$ (nullary symbols)

then $f_A$ is monotone w.r.t. $>_0$
Exercise: ... and this does not hold if any of the conditions are violated.

## Exotic Multilinear Functions and Compatibility

how to check that exotic matrix int. is compatible with rewrite rule?
By previous slide, it is enough to consider $f_a(x_1^T) = M_1 x_1^T$
Also recall that top-left$(M_1) \neq 0_E$.
Define $>_0$ on $E^{d \times d}$ by point-wise extension of $>_0$ on $E$.
Prop. $A >_0 B$ implies $Ax^T >_0 Bx^T$ for $x$ with $x_1 \neq 0_E$.
example: $aa \to aba$ tropical with dim 3

## Exotic Interpretations and Complexity

Thm. If $R$ admits an arctic or tropical matrix interpretation, then $\mathrm{dc}_R$ is linear.

# Top-Termination

Power of exotic matrix interpretations is much increased in combination with:
Dependendy Pairs Transformation:

- reduces a termination problem SN($R$)
- to a relative top-termination problem SN(DP($R$)$_{\text{top}}/R$)

because

- for top termination, interpretation need not be monotone at all
- for the "relative" part, need only be weakly monotone

# Fuzzy Matrix Interpretations

the (min,max) semiring
since addition (min) is not monotone: remarks on arctic and tropical do apply: method cannot handle symbols of arity $> 1$
but does it work for unary? No:

- multiplication (max) is again not monotone.
- there are not enough values (height of interpretation is finite)

but

- this can be repaired (by transforming to tropical matrices)
- there is an efficient semi-algorithm to find fuzzy matrix int.

# From Fuzzy to Tropical

consider only unary matrix interpretations of shape $f_A(x^T) = M \cdot x^T$
Def: from fuzzy matrix $F$, compute tropical matrix $T = \text{lift}_d(F)$
by point-wise multiplication by $d$.
Thm: Let $A$ be a fuzzy matrix interpretation compatible with SRS $R$.
Let $m =$ max. finite entry in matrices in $A$ and $w =$ max. length of rhs of $R$.
Then $\text{lift}_{m \cdot w}(A)$ (point-wise) is a tropical interpretation that is compatible with $R$.

# (Fuzzy) Matrix I. as Weighted Autom.

given fuzzy matrix int. $A$ compatible with SRS $R$, consider automaton (graph) on states $Q = \{1, \ldots, d\}$, with 1 initial and final,
and edge $p \overset{(a,h)}{\to} q$ iff $M_a(p,q) = h < +\infty$.
concatenate $p \overset{(a_1,h_1)}{\to} q \overset{(a_2,h_2)}{\to} r$ to $p \overset{(a_1 \cdot a_2, \max(h_1,h_2))}{\to} r$.
this automaton computes a function
$A : Q \times \Sigma^* \times Q \to F : (p,w,q) \mapsto \min\{h | p \overset{(w,h)}{\to} q\}$
compatibility means:
$\forall (l,r) \in R, p,q \in Q : A(p,l,q) >_0 A(p,r,q)$.
construction of compatible interpretation $\sim$ completion of automaton

# Match-bounded string rewriting

for $R$ over $\Sigma$, define $\text{match}_h(R)$ over $\text{lift}(\Sigma)$ by $\{l' \to \text{lift}_{h-1}(r) \mid \text{base}(l') = l, \max\text{height}(l) = h\}$
Ex.
$\text{match}_3(aa \to aba) = \{\ldots, a_3 a_0 \to a_2 b_2 a_2, \ldots\}$
For $R$-rewrite sequence $s : w_0 \to w_1 \to \ldots$
obtain $\text{match}_h(R)$-sequence $s'$: start with $\text{lift}_h(w_0)$ and apply lifted rule at position.
Def: $R$ is *match-bounded by h* if each $s$ has a corresponding $s'$ (i.e., labels do not go below 0).
Prop. $R$ is match-bounded by $h \iff R$ admits a compatible fuzzy matrix interpretation with entries $\leq h$.

# A Decomposition Result

new letters $\overleftarrow{a}$, $\overrightarrow{a}$,
operation on words $\overleftarrow{ab} = \overleftarrow{b}\overleftarrow{a}$, etc.
define
$E = \{a_x \overleftarrow{a_x} \to \epsilon \mid a \in \Sigma\} \cup \{\overrightarrow{a_x} a_x \to \epsilon \mid a \in \Sigma\}$
from each rule $(p a_h q \to r) \in \text{match}_h(R)$, where $a_h$ is largest letter, construct rule $a_h \to \overleftarrow{p}\, r\, \overrightarrow{q}$,
obtain set of rules $C$.
Prop. $\to_{\text{match}(R)} = \to_C \circ \to_E^* \cap (\text{lift}(\Sigma)^*)^2$
Thm. $\to_{\text{match}(R)}^* = \to_C^* \circ \to_E^* \cap (\text{lift}(\Sigma)^*)^2$
Note: $\to_C$ is subsitution, $\to_E$ is monadic, both are effectively REG-preserving
Cor. match($R$) is REG-preserving
Cor. $R$ match-bounded $\Rightarrow$ $R$ REG-preserving

# A completion procedure

- start with 1 state and loops $1 \overset{(a,h)}{\to} 1$
- while there is $(l,r) \in R, p,q \in Q$ such that $h = A(p,l,q) \not>_0 A(p,r,q)$:
  - pick $p', q' \in Q$ such that $p' \overset{(a,h)}{\to} q'$ is on path from $p$ to $q$, where $l = u \cdot a \cdot v$.
  - add fresh path from $p'$ to $q'$ labelled $\overleftarrow{u} \cdot \text{lift}_{h-1}(r) \cdot \overrightarrow{v}$ (this is one $\to_C$-step)
  - close the automaton w.r.t. $\to_E$ (by adding $\epsilon$-transitions)

Thm. this construction terminates iff $R$ is match-bounded by $h$, the result is a fuzzy-weighted automaton compatible with $R$

# Decision Problems for Exotic Int.

for $E$ in natural, arctic, fuzzy semiring, consider these decision problems:

- given $R$ (and $d$, and $h$), does $R$ admit a comaptible $E$-interpretation with matrix size $\leq d$ and entries $\leq h$?

known results about complexity:

- $d$ and $h$ given: decidable (by enumeration)
- $d$ given: decidable for fuzzy (can derive a bound for $h$)
- $h$ given: decidable for fuzzy (by automata completion)