Exercises in Termination

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1 Derivational Complexity

1.1 Three rewriting systems that look alike

Find lower bounds for the derivational complexity of:

- $R_1 = \{ba \to acb, bc \to abb\}$
- $R_2 = \{ba \to acb, bc \to cbb\}$
- $R_3 = \{ba \to aab, bc \to cbb\}$

Hint: one is doubly exponential, one is multiply exponential, one is non-terminating.

A lower bound is proved by presenting a family of derivations that achieves the desired length.

1.2 How To Count

(H. Zantema) Give an example of a length-preserving string rewriting system that has exponential derivational complexity, by simulating a binary counter. Use letters 0,1 and a "carry" symbol.

Hint: the "obvious" solution has total size (sum of lenghts of all lhs and rhs) 12. Can you do better?

- show exactly a family of derivations of exponential length
- give a termination proof

1.3 How to Count (with size 8)

Same problem as above. Prove that this works: $\{a \to b, abb \to baa\}$.

1.4 How to Count with one length-preserving rule

Can we get exponential derivation lengths with one-rule length-preserving? (communicated by A. Geser) No, Y. Metivier (TCS 35(1985) 71–78) proves that one-rule length-perserving has polynomial derivation lengths, and A. Bertrand (TCS 123(1994) 21–30) proves that the polynomial is quadratic, so $ab \to ba$ is essentially the worst that can happen.

What if it's non-terminating? Prove that it is decidable whether a one-rule lenght-presering SRS is terminating. (Hint: the decision procedure is *tres simple*.)

Generalize to one-rule size-preserving or depht-non-increasing TRSs.

1.5 Such A Loong Looop

(A. Geser) Find a looping derivation for $R_k = \{10^k \to 0^k 1^k 0\}$.

1.6 All (Natural) Degrees

(D. Hofbauer) We choose a number $d \in \mathbb{N}$ with $d \geq 2$ and consider the system $R_d = \{xy \to zx \mid z < x\}$ over $\Sigma_d = \{1, \ldots, d\}$.

Show that the derivational complexity of R_d is $\Theta(n^d)$.

Of course, this requires two proofs:

- there is a derivation of the given length
- there is no longer derivation

Hint (for both parts): the x symbol (the largest in each rule) moves from left to right.

Extra question: find the smallest subset of R_d that still has the same properties.

1.7 In Between Days

Find a string or term rewriting system whose derivational complexity is polynomially bounded, but not big-Theta of any polynomial. E.g. construct a system of complexity $n \mapsto n \cdot \log n$.

1.8 Some Unnatural Degrees

(H. Zantema) Find a rewriting system with derivational complexity $\Theta(n^d)$ where d is not an integer.

Hint: for instance, $d = \log_2(3)$.

Can you do with one rule only?

1.9 Quadratic used to be Easy

Give an example of a string rewriting system with quadratic derivational complexity where all rules are length-increasing.

(So the obvious $ab \to ba$ is ruled out.)

2 Match Bounds

2.1 Right. Forward!

Verify that for $k \geq 2$, the system $R_k = \{baba^k \to a^{k+1}babab\}$ is RFC-matchbounded by 2. Construct the automaton, it is not too hard.

Note: this problem is treated (without match-bounds) in Section 6.6 of Geser's Habil thesis.

2.2 How High Can You Get

For any $n \in \mathbb{N}$, construct a SRS R_n that is exactly match-bounded by n.

- (easy) number of rules and size of alphabet may depend on n
- alphabet is fixed (but number of rules may depend on n)
- (hard) both alphabet and number of rules is fixed (so, length of rules depends on n)

2.3 Match Left and Match Right

(research problem)

We define a method to annotate positions in strings with heights: in a rule application $l \to r$ with $|l| > 1$ and $|r| > 1$, choose any nontrivial representation $l = l_1 l_2, r = r_1 r_2$, and then annotate with match heights for $l_1\ tor_1$ and $l_2 \to r_2$ separately (the r_1 part gets $1 + \min l_1$, the r_2 part gets $1 + \min l_2$). Any derivation annotated in that manner is called a split-match derivation. Prove that a system is split-match-bounded iff it is match-bounded.

2.4 Near Ground Level (Plain)

Determine the class of one-rule string rewriting systems that are match-bounded by one. (Discuss what overlaps are allowed between lhs and rhs.)

2.5 Near Ground Level (RFC)

Determine the class of one-rule string rewriting systems that are RFC-matchbounded by zero (i.e. they have no R-redex in the RFC closure automaton, assuming that the "extension rules" produce match height zero).

Here is an example: $R = \{abaa \to a^3bbab\}$. We compute $\text{RFC}(R) = (a^3b^2)^*ab$ and this contains no R-redex.

2.6 The size without the bound

Is the following questions decidable: does a finite string rewriting system R admit a match-bound certificate

- with given size (number of states) s and given bound b?
- with given bound (but any size)?
- with given size (but any bound)?
- with no further information? Remark: if you can choose the starting language, then this is undecidable (Middeldorp), but here the question is for Σ^*.

3 Matrix Method(s)

(we restrict to the "standard" matrix method (shape E) where top left and bottom right entry of matrices are positiv, and top right entry of difference is positive.)

3.1 Small Matrices

Give the smallest matrix proof dimension for these systems:

- $R_1 = \{aa \to bbb\}$
- $R_2 = \{ab \to ba\}$
- $R_3 = \{aa \to aba\}$.

Hint: one, two, three.

The easy part is (probably) to find the interpretation, the harder part is to prove that no smaller interpretation exists. You need to prove the theorem that matrix interpretations of dimensions ≤ 2 in fact show *simple* termination.

3.2 All Degrees

(see exercise in complexity)

Show that assigning weight $\binom{p}{k}$ to letter k standing at position p (counting from the right) gives a compatible interpretation (the sum of weights is decreasing).

Show that this weight can be computed by a matrix interpretation. Hint: the matrices are upper triangular.

3.3 A Path Is Enough

Characterize those one-rule SRS that have a matrix interpretation that just consists of the path corresponding to the left-hand side, all edges labelled by one. (And of course the Σ-loops at start and end.)

3.4 Just One Bit Of Information

(research problem) Characterize (some of) those systems that admit a matrix interpretation where all entries in all matrices (interpretations of symbols and rules) are in $\{0, 1\}$. (This is what would happen if you restrict the bit width in the SAT translation to one.)

3.5 Some Calculus

(Using some computer algebra system,) take any matrix interpretation (e.g. the 5-dimensional for z001), pick one position and replace the entry there by a variable. Then discuss (plot) the difference(s) between lhs and rhs interpretations as functions of that variable.

3.6 Polynomially Growing Matrix Interpretations

(research problem) For $a^2b^2 \to b^3a^3$ (Zantema's Problem) and $\{a^2 \to bc, b^2 \to ac, c^2 \to ab\}$ (Zantema's Other Problem), we have compatible 5-dimensional matrix interpretations (over $\mathbb{N}$) but they grow exponentially.

It is known that ZP is linear (e.g., because it is match-bounded). It is known that ZOP is quadratic (S. I. Adian, Mathematical Notes July 2012, Volume 92, Issue 1-2, pp 3-15).

For each of these systems, find matrix compatible interpretations that (a) grow polynomially, (b) ... with the right degree — or prove that none exist.

4 Automated Termination

4.1 That's a Classic

SRS/Zantema/z001 ... z128 is a classical problem set in termination of string rewriting. E.g. z001 is Zantema's Problem $\{a^2b^2 \to b^3a^3\}$, Without looking into your computer or the internet, answer the following:

- how many problems are these? (Hint: some are missing from the enumeration.

  Bonus points: when (what year) were they removed and why?)

- what number is Zantema's Other Problem $\{a^2 \to bc, b^2 \to ac, c^2 \to ab\}$? When was it first solved? By what program, using what method?

  (Bonus points: why is this proof not visible in the results table?)

- what problems are looping?

- what problem is non-looping non-terminating?

- what problem is Ackermannian? was it solved?

4.2 Common Knowledge

What is the common property of SRS/Gebhardt/* (besides being hard)?

4.3 More Common Knowledge

The common property of ICFP2010/* is that they all come from the same source (in fact they were generated over one weekend, by a lot of people) and all have a termination proof by a special type of matrix interpretation. What source? What shape?