Termination and Complexity
Notations for Rewriting
Intl. School on Rewriting 2014

Johannes Waldmann (HTWK Leipzig)

August 25, 2014

Signature, Terms

signature: set of function symbols with arities
example: $\Sigma = \{f/2, S/1, Z/0\}$.
term t over signature Σ: $t = f(t_1, \ldots, t_k)$ where
$(f/k) \in \Sigma$ and $\forall 1 \le i \le k : t_i \in \mathrm{Term}(\Sigma)$
example: $f(S(Z()), Z()) \in \mathrm{Term}(\Sigma)$.

Positions, Subterms

positions in t:
$\mathrm{Pos}(f(t_1, \ldots, t_k)) = \{[]\} \cup \{[i] \circ p \mid 1 \le i \le k, p \in \mathrm{Pos}(t_i)\}$
example:
$\mathrm{Pos}(f(S(Z()), Z())) = \{[], [1], [1, 1], [2]\}$
subterm at position: for $p \in \mathrm{Pos}(t)$:
$t([]) = t, f(t_1, \ldots, t_k)([i] \circ p) = t_i(p)$
function symbol at position: for $p \in \mathrm{Pos}(t)$:
$f(\ldots)([]) = f, f(t_1, \ldots, t_k)([i] \circ p) = t_i(p)$
(note: overloaded notation)

Positions, Replacements at position

applications:
subterm relation: $s \trianglelefteq$ *tiff* $\exists p \in \mathrm{Pos}(t) : s = t(p)$
size of term: $|t| = |\mathrm{Pos}(t)|$
$t(p := s')$ for $p \in \mathrm{Pos}(t)$:
$t([] := s) = s, f(\ldots, t_i, \ldots)([i] \circ p := s) = f(\ldots, t_i(p := s), \ldots)$.

Variables, Substitutions

term over signature Σ with variables from set X
(disjoint from Σ): $\mathrm{Term}(\Sigma, X)$:
$x \in X \Rightarrow x \in \mathrm{Term}(\Sigma, X)$ and
$\forall i : t_i \in \mathrm{Term}(\Sigma, X) \Rightarrow f(t_1, \ldots, t_k) \in \mathrm{Term}(\Sigma, X)$
ground substitution: partial mapping
$\sigma : X \hookrightarrow \mathrm{Term}(\Sigma)$
extended to mapping
$\sigma : \mathrm{Term}(\Sigma, X) \to \mathrm{Term}(\Sigma)$, written post-fix
by $x\sigma = \sigma(x), f(t_1, \ldots, t_k)\sigma = f(t_1\sigma, \ldots, t_k\sigma)$

Term Rewriting

rule $(l, r) \in \mathrm{Term}(\Sigma, X)^2$
apply rule (l, r) at position p in term t
$t \overset{(l,r),p}{\to} t'$ if

- $p \in \mathrm{Pos}(t)$,
- $\exists \sigma : t(p) = l\sigma \land t(p := r\sigma) = t'$

rewrite system R is set of rules
$t \to_R t'$ iff $\exists (l, r) \in R, p : t \overset{(l,r),p}{\to} t'$

String Rewriting as a special case

- all symbols are unary
- there is one nullary symbol “0” that never occurs in rules

example $R = \{a(b(x)) \to b(a(x))\}$ over signature $\Sigma = \{a/1, b/1, 0/0\}$

- terms are in fact strings: $a(b(a(b(0)))) \sim abab \in \{a, b\}^*$
- a rule is a pair of strings, $a(b(x)) \to b(a(x) \sim ab \to ba$
- string rewriting system $R \subseteq (\Sigma^*)^2$
- string rewriting relation: $w \to_R w' \iff \exists (l, r) \in R, p, q \in \Sigma^* : w = plq, prq = w'$

Summary

- these are basic concepts, cf. also Baader/Nipkow
- you should work on the online exercises that require you to find derivations
- there is a highscore ranking: longer derivations are better (this is preparation for derivational complexity)