Size-Change Termination and Arctic Matrix Monoids

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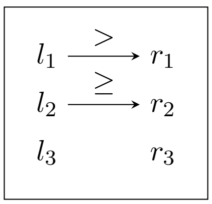
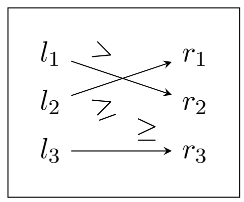
In this note, we show a connection between a boundedness problem for matrix products over the arctic semi-ring $(\{-\infty\} \cup \mathbb{N}, \max, +)$ and the "size-change" method for proving termination of programs [3].

This method translates an input program R into a set of size-change graphs that represent relations between argument sizes in function calls. For each rule $f(l_1, \ldots, l_m) \to t$ in R, and for each subterm $g(r_1, \ldots, r_n)$ of t, a directed, labelled, bipartite graph with m input vertices and n output vertices is constructed that contains edges $l_i \to r_j$ with label $>$ or $\geq$, expressing strong or weak decrease of the argument size from l to r in the corresponding positions.

For instance, the program rule $F(x+1, y+1, z) \to F(x, y+1, F(y+1, x, z))$ is translated to two size-change-graphs (one for every occurrence of F in the right-hand-side of the rule) given in the boxes:



$$\begin{pmatrix} 1 & -\infty & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \end{pmatrix}$$



$$\begin{pmatrix} -\infty & 1 & -\infty \\ 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}$$

We observe that every size-change-graph can be represented as matrix with entries in $\{-\infty, 0, 1\}$ where 0 represents $\geq$-labelled edges, 1 represents $>$-labelled edges, and $-\infty$ represents non-existent edges in G.

A sequence of nested function calls corresponds to a composition of size-change graphs, and multiplication of matrices in the arctic semi-ring, respectively. If each infinite sequence of size-change graphs contains a path with infinitely many $>$-edges, then the program terminates by contradiction to the well-foundedness of the parameter domain (in the example, $\mathbb{N}$).

The corresponding property for products of arctic matrices from a set S is that each infinite sequence has a suffix for which norms of products of prefixes are unbounded:

Definition 1. A sequence $(G_i)_{i<\omega} \in S^\omega$ is *universally tail-unbounded* if there is some k such that $\sup\{\| \prod_{i=k}^{k'} G_i \| \mid k' \geq k\}$ is infinite, where $\|M\|$ is the maximal entry of M.

In general, not all sequences of matrices are meaningful for the program. Consequently, S^ω needs to be refined as the set of infinite paths through a finite automaton that has subprogram names as states, and transitions are labelled by matrices. This is similar to meta-transitions in [2].

For the problem of deciding universal tail-unboundedness, result from the size-change literature can be translated. The problem is PSPACE-complete [3], and a generalization to arctic integers $(\{-\infty\} \cup \mathbb{Z})$ is undecidable, but becomes again decidable for "fan-in one" (each matrix column contains at most one finite entry) [1]. For proving program termination automatically, it is of practical interest to find more (and efficiently) decidable restrictions of the problem.

References

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[2] Daniel Kirsten and Sylvain Lombardy. Deciding unambiguity and sequentiality of polynomially ambiguous min-plus automata. In Susanne Albers and Jean-Yves Marion, editors, *STACS*, volume 3 of *LIPIcs*, pages 589–600. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany, 2009.

[3] Chin Soon Lee, Neil D. Jones, and Amir M. Ben-Amram. The size-change principle for program termination. In *POPL*, pages 81–92, 2001.