Match-Bounds for Relative Termination

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Match-Heights and -Bounds

- annotate positions by numbers (go from Σ to $\Sigma \times \mathbb{N}$)
- start with all zeroes. in each step: heights in reduct $:=$ 1 + min of heights in redex

Def: R is match-bounded by b on $L \subseteq \Sigma^*$: each R-derivation starting in L has all heights $\leq b$.

- R m.b. on $L \Rightarrow R$ is terminating on L.
- R m.b. $\Rightarrow R$ effectively preserves REG.

Note: min heights in redex $<$ min heights in reduct

Fuzzy Matrix Interpretations

$\mathbb{F}$ = the fuzzy semiring on $\mathbb{Z} \cup \{-\infty, +\infty\}$,

- addition = "min", zero = $+\infty$,
- multiplication = "max", one = $-\infty$.

orders:

- natural order $>$ on $\mathbb{F}$,
- $x >_1 y$ is $(x = +\infty = y) \vee (x > y)$,
- extend $>_1$ point-wise to matrices.

$\mathbb{F}$-matrix interpretation $[\cdot]$ is compatible with R if $[l] >_1 [r]$ for all rules $(l \to r) \in R$.

Support of an Interpretation

Support of $[\cdot]$: all w with $[w] \neq 0$
(note: zero matrix has $+\infty$ everywhere)
Thm: If R admits compatible $[\cdot]$,

- then R is match-bounded for $\operatorname{supp}[\cdot]$,
- and $\operatorname{supp}[\cdot]$ is closed w.r.t. $\to_R$.

Proof ideas:

$\mathbb{F}$-interpretation corresponds to match-bound certificate automaton: for each redex path E, there is a reduct path U with $\min \operatorname{height} E < \min \operatorname{height} U$

zero ($+\infty$) is highest in the order

Fuzzy Interpretation Ex.

$$a \mapsto \begin{pmatrix} 3 & 0 & \infty \\ \infty & \infty & \infty \\ 2 & 0 & \infty \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 2 & 3 & 1 \\ \infty & \infty & 1 \\ \infty & 3 & 1 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 0 & 3 \\ \infty & 0 & 6 \\ \infty & 0 & \infty \end{pmatrix}, \quad d \mapsto \begin{pmatrix} 4 & 4 & \infty \\ \infty & \infty & \infty \\ \infty & \infty & \infty \end{pmatrix}$$

$\forall x \in \Sigma : [x]_{\text{top,left}} < +\infty$, thus $\text{supp}[\cdot] = \Sigma^*$.

is compatible with (SRS/Zantema/z003):
$\{bca \to ababc, b \to cc, cd \to abca, aa \to acba\}$

Fuzzy Interpr. and Algebras

what is the underlying monotone algebra?
vectors of multisets of heights,
equivalently, of tropical numbers.

Relative Termination

R is terminating relative to S
if $\mathrm{SN}(R/S)$ where $R/S = S^* \circ R \circ S^*$
(any $R \cup S$-derivation has only finitely many R-steps)
Ex.: $\mathrm{SN}(\{aa \to aba\}/\{b \to bb\})$.
Application: $\mathrm{SN}(R/S)$ and $\mathrm{SN}(S)$ imply $\mathrm{SN}(R \cup S)$.

Relative Match-Bounds

Def: (R, S) is match-bounded by b for L, if in each annotated mixed derivation that starts in L, each R-rule has: min height redex $< b$.

Thm: If (R, S) is match-bounded, then $\mathrm{SN}(R/S)$.

Proof: consider multiset of heights $< b$

idea for certificates/interpretations:
unify all heights $\geq b$.
since order is reversed: use $-\infty$.

Fuzzy Int. for Relative Term.

use $>_2$ on $\mathbb{F}$ (and point-wise on matrices) where $x >_2 y$ iff $(x = +\infty = y) \vee x >_1 y \vee (x = -\infty = y)$.

$\mathbb{F}$-matrix interpretation $[\cdot]$ is weakly compatible with S if $[l] >_2 [r]$ for all rules $(l \to r) \in S$

If a fuzzy interpretation $[\cdot]$ is compatible with R and weakly compatible with S, then (R, S) is match-bounded on $\mathrm{supp}[\cdot]$.

If both (R, S) and S are match-bounded, and $\mathrm{rhs}(R)$ and $\mathrm{lhs}(S)$ are overlap-free, then $R \cup S$ is match-bounded.
(non-overlap condition is essential)

Comparison with Zankl and Kor

(RTA 2010): for redexes of relative rules that are non-size-increasing, and have all lhs labels equal, the rhs gets the exact same label.

notions are incomparable (even for string rewriting):

- $\{aa \to aba\}/\{b \to bb\}$
- $\{a \to b\}/\{a \to a\}$

Forward Closures

$\mathrm{RFC}(R) =$ right-hand sides of R-forward-closures.
R terminating $\iff$ R terminating on $\mathrm{RFC}(R)$.

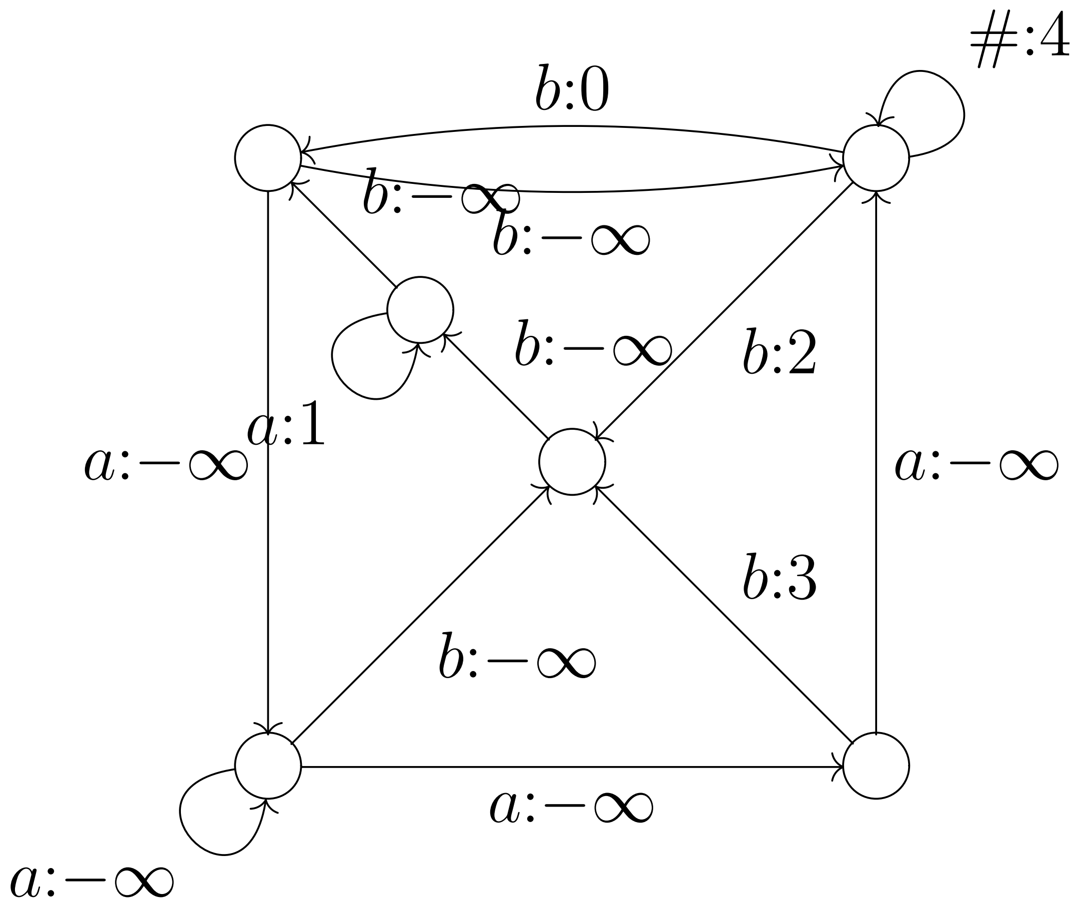
simulate narrowing by rewriting:
$\mathrm{RFC}(R)\#^* = (R \cup R_\#)(\mathrm{rhs}(R)\#^*)$, where
$R_\# = \{(u\# \to r) \mid (uv \to r) \in R, u \neq \epsilon \neq v\}$.

find $[\cdot]$ that is comp. with R and weak comp. with $R_\#$ and $\mathrm{rhs}(R)^* \subseteq \mathrm{supp}[\cdot]$.

ansatz: $[\#]_{\text{top,left}} = -\infty, [\#]_{\text{elsewhere}} = +\infty$
check weak compat. with $R_\#$ by looking at leftmost columns of l, u.

Forward Closures Ex.

For $\{a^2b^2 \to b^3a^3\}$, RFC match-bound certificate:



RFC and Relative Termination

cannot prove relative termination (on Σ^*) by restricting to RFC:

- $R = \{ab \to a\}, S = \{c \to bc\}$
- $\mathrm{RFC}(R \cup S) = \{a\} \cup b^*c$.
- $\mathrm{SN}(R/S)$ on $\mathrm{RFC}(R \cup S)$, since $\mathrm{RFC}(R \cup S)$ does not contain any R-redex.
- not $\mathrm{SN}(R/S)$ on Σ^*, since $abc \to_R ac \to_S abc$

RFC and Rule Removal

can still apply the RFC method to “remove rules” in a modular proof of full termination:

Thm: If $\mathrm{SN}(R/S)$ on $\mathrm{RFC}(R \cup S)$ and $\mathrm{SN}(S)$ on Σ^*, then $\mathrm{SN}(R \cup S)$ on Σ^*.

For $R = \{cb \to bbc\}$ and $S = \{ab \to baa\}$:

we have $\mathrm{RFC}(R \cup S) \subseteq \{a, b\}^* \cdot \{c, \epsilon\} = L$,

L contains no R-redex:
(R, S) is RFC-matchbounded by 0,
thus $\mathrm{SN}(R/S)$ on $\mathrm{RFC}(R \cup S)$

get $\mathrm{SN}(S)$ from reversal and RFC.

Discussion

- similar thing works for tropical interpretations
- verification of
  - match-bounds / fuzzy interpr.
    easiest: by transformation to . . .
  - tropical interpr. (Adam Koprowski, Color)

  probably need sparse matrix representation
- other semirings with zero at the top?
  (and superlinear growth?)