Tropical Termination

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Definition:

- domain: $\mathbb{T} = \mathbb{N} \cup \{+\infty\}$,
- operations:
  - $\oplus$ (semiring addition): min
  - $\otimes$ (semiring multiplication): plus
- $\mathbb{T}$-weighted automaton over Σ
  $\approx$ directed graph, labelled with $\Sigma \times \mathbb{T}$

Applications

- Imre Simon: used $\mathbb{T}$-weighted automata to analyze *finite power property* and *star height* of regular languages
- shortest path problems/algorithms (e.g., Dijkstra's) use $\mathbb{T}$-weighted automata (often implicitly)
- tropical (algebraic) geometry, tropical analysis

Tropical Order

- Definition: $x \geq y \iff x \oplus y = y$
  has the usual nice properties since tropical addition is idempotent.
- Properties of the strict part $>$:
  - monotone w.r.t. multiplication (except at zero)
  - not monotone w.r.t. addition:
    $5 > 4$ but $(5 \oplus 3) \not> (4 \oplus 3)$
  - $a > b$ and $c > d$ imply $(a \oplus c) > (b \oplus d)$
- will treat zero in a special way, define
  $x >_0 y \iff (x > y) \vee (x = +\infty = y)$
- will use point-wise product of $\geq$ (and $>_0$, resp.) on $\mathbb{T}$-vectors.

Matrix Interpretations

- for a term rewriting system R over Σ,
  and semiring W (natural, arctic, tropical, ...)
  map function symbols to linear vector-valued function

$$[f](x_1, \ldots, x_k) = V_0 \oplus M_1 \otimes x_1 \oplus \ldots \oplus M_k \otimes x_k$$

- obtain an *extended weakly monotone algebra* (Zantema)
  - orders $>, \geq$, compatible
  - $[f]$ is monotone w.r.t. $\geq$
  - ($[f]$ not necessarily monotone w.r.t. $>$)

  If $[l] > [r]$ for rules of R, and $[l] \geq [r]$ for rules of R',
  and $>$ well-founded,
  then R is top-terminating relative to R'.

Tropical Matrix Interpretations

- domain: $\mathbb{N} \times \mathbb{T}^k$
- orders on domain:
  - $\geq$: point-wise extension of $\geq$ on $\mathbb{T}$
  - $>$: point-wise extension of $>_0$
- orders on interpretations (again, point-wise extensions)
  - $[f] > [f'] \iff V_0 > V_0' \wedge M_1 > M_1' \wedge \ldots \wedge M_k > M_k'$
  - then $f(x_1, \ldots, x_k) > g(x_1, \ldots, x_k)$
- notes: well-founded, stay inside the domain ($[f]$ is "somewhere finite")
- formally verified in Coq (as part of Color) by Adam Koprowski
- easy analogy to arctic matrix interpretations.
- common "exotic" framework? (cf. Thiemann PR2010)

Exotic Full Termination

- (that is, not just top termination)
- can only work for unary signatures (string rewriting):
  assume $t_1 \to_R t_2$ and $[t_1] > [t_2]$.
  then $[f(t_1, s)] = V_0 \oplus M_1[t_1] \oplus M_2[s]$
  but $\oplus$ is not monotone w.r.t. $>$.
  absolute parts of interpretations must be zero.
- then, interpretation is mapping $\Sigma \to \mathbb{T}^{d \times d}$
  matrices are orderd by point-wise $>_0$, as before.
  "somewhere finite" means top-left entry is finite.

Match-Bounds

- Theorem: from each match-bound certificate automaton for R, we can construct a tropical interpretation that is compatible with R.
- Proof idea: "matchbounded $\Rightarrow$ terminating" uses multisets
  This can be replaced by weights (because of bounded branching), these weights *are* tropical numbers.
- example: match-heights: $a_3 a_0 \to a_1 b_1 a_1$
  assume match-bound is 3, and
  $\max \text{length} \operatorname{rhs}(R) = 3, \min \text{length} \operatorname{lhs}(R) = 2$:
  map heights to weights: $3 \mapsto 0, 2 \mapsto 1, 1 \mapsto 2, 0 \mapsto 4$
- Application: formal verification of match-bounded proofs can be done by transformation to tropical proofs
- but probably need sparse matrix representations

Local Termination

- Definition: R over Σ is (locally) terminating on $L \subseteq \Sigma^*$:
  there is no infinite R-derivation that starts in L.
- one way to prove local termination
  (de Vrijer, Endrullis, W.: RTA 09)
  - find a partial model = a (classical) automaton A over Σ (deterministic, but not necessarily complete) that contains L and is locally R-closed,
  - prove full termination of "R labelled by states of A"
- tropical interpretations can do both steps at the same time (roughly speaking)
  ... same idea as for match-bound certificates

- for any tropical matrix interpretation $i : \Sigma \to \mathbb{T}^{d \times d}$, define its *support*

$$\mathrm{supp}(i) = \{ w \mid w \in \Sigma^*, i(w) \neq 0 \}$$

- observation: $\mathrm{supp}(i)$ is a regular language
Proof: use homomorphism from $\mathbb{T}$ to $\mathbb{B}$
- observation: if i is compatible with R, then $\mathrm{supp}(i)$ is closed w.r.t. $\to_R$.
Proof: trace paths in i, and use that zero ($= +\infty$) is maximal.
- Theorem: if i is compatible with R, then R is locally terminating on $\mathrm{supp}(i)$.

- Forward Closures of R: obtained from R by right extension (narrowing) and rewriting.
- $\mathrm{RFC}(R)$ = right hand sides of R-forward-closures.
- Theorem: R is terminating on Σ^* iff R is locally terminating on $\mathrm{RFC}(R)$.
- observation: use rewrite-implementation of extension

$$R_\# = \{ u\# \to r \mid (u \cdot v, l) \in R, u \neq \epsilon \neq v \}$$

to compute

$$\mathrm{RFC}(R)\#^* = (R \cup R_\#)^*(\mathrm{rhs}(R)\#^*)$$

Tropical RFC Termination

Theorem: if a tropical interpretation i

- is compatible with $R \cup R_\#$
- and $\mathrm{rhs}(R)\#^* \subseteq \mathrm{supp}(i)$,

then R is terminating.

simplified realization (ignoring the interpretation of $\#$):

- assume the only $\#$-transition in i is a loop at state q
- assume the weight of this transition is large (but finite)
- for each $(l, r) \in R$, there must be some r-path to q
- for each $(u \cdot v, r) \in R$ with $u \neq \epsilon \neq v$, if there is a u-path from p to q, then there must be an r-path from p to q

Tropical Constraints

find compatible interpretation as solution of a constraint system (for the matrix entries).

- general observation: the system is a boolean combination of linear inequalities: satisfiability is decidable (for fixed dimension).
- booleans can be simulated by integers (mixed linear programming)
but we have a lot of booleans (one or each $\oplus$)
- or we restrict the range and make a finite domain problem that can be solved by bit-blasting.
different choices for representation:
  - binary
  - unary (note: addition via sorting networks)
  - ad-hoc CNFs for small fixed bit widths

Status, Outlook

- basic idea (full tropical termination) published in WATA06, JALC 2007 (historic note: JALC $\leftarrow$ EIK)
- tropical termination (full, top, RFC): implemented in Matchbox (2010)
- performance:
in some cases, finds smaller certificates than other methods,
still missing: "killer examples"
- verification (tropical top termination): implemented in Coq/Color by Adam Koprowski
- "relative RFC termination" in WST 2010 (with Dieter Hofbauer)
(for match-bounds, but same thing works for tropical)