Matrix Evolutions

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Integer Matrix Interpretation: Example

Example (SRS/Zantema/z057)

$\mathcal{R}_{z057} = \{babbab \to ababba\}$

$$[a] = \begin{pmatrix} \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}, \quad [b] = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix},$$

$$[babbab] - [ababba] = \begin{pmatrix} 0 & 0 & 2 & 0 & \boxed{2} \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Integer Matrix Interpretation: Definitions

Definition (Domain, Range)

$$E = \{A \in \mathbb{N}^{n \times n} \mid A_{1,1} \geq 1 \wedge A_{n,n} \geq 1\}$$
$$P = \{A \in \mathbb{N}^{n \times n} \mid A_{1,n} > 0\}$$

Definition (Matrixorder >)

$$A > B :\Leftrightarrow A - B \in P$$

Remark

$$\forall A, B \in E (A > B \Leftrightarrow \forall i, j \in \{1 \ldots n\} : A_{i,j} \geq B_{i,j} \wedge A_{1,n} > B_{1,n})$$

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Integer Matrix Interpretation: Soundness

Theorem (Hofbauer, Waldmann 2006)

If there exists an interpretation $[\cdot] : \Sigma \to E$ *such that for every rule* $l \to r$ *of a string rewriting system* $\mathcal{R}$

$$[l] > [r]$$

holds, then $\mathcal{R}$ *is terminating.*

Non-Integer Matrix Interpretation: Example

Example (SRS/Zantema/z057)

$\mathcal{R}_{z057} = \{babbab \to ababba\}$

$$[a] = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}, \quad [b] = \begin{pmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix},$$

$$[babbab] - [ababba] = \begin{pmatrix} 0 & 98/3^4 & 5/3^5 & \boxed{1/3^3} \\ 0 & 4/3^5 & 116/3^7 & 40/3^5 \\ 0 & 2/3^5 & 8/3^5 & 208/3^5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Non-Integer Matrix Interpretation: Definitions

Definition (Domain, Range)

$$E' = \{A \in \mathbb{R}^{n\times n} \mid A_{i,j} \geq 0 \wedge A_{1,1} \geq 1 \wedge A_{n,n} \geq 1\}$$
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Definition ((strict) Range)

$$P'_d = \{A \in \mathbb{R}^{n\times n} \mid A_{i,j} \geq 0 \wedge A_{1,n} > d\}$$

compute d as $\min\{([l] - [r])_{1,n} \mid (l \to r) \in \mathcal{R}\}$

Definition (Matrixorder $>_d$)

$$A >_d B :\Leftrightarrow A - B \in P'_d$$

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Non-Integer Matrix Interpretation: Soundness

Theorem (Gebhardt, Hofbauer, Waldmann 2007)

If there exists an interpretation $[\cdot] : \Sigma \to E'$ *and* $d > 0$ *such that for every rule* $l \to r$ *of a string rewriting system* $\mathcal{R}$

$$[l] >_d [r]$$

holds, then $\mathcal{R}$ *is terminating.*

Non-Integer proof of rel11

- Competition 2006: TPA ($2 \times$ labeling, polynomial interpret.)
- Competition 2007: no proof

Example (SRS/Zantema06/rel11)

$$\mathcal{R}_{rel11} = \{bpb \to bapb,\ p \to= apa,\ apaa \to= p\}$$

$$[a] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ [b] = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ [p] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[\mathcal{R}] = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1-x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1-xy & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & xy^2-1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

valid proof if $0 < x < 1 \wedge \sqrt{\frac{1}{x}} \leq y \leq \frac{1}{x}$,
e.g. $x = \frac{1}{2}$, $y = 2$

Non-Integer proof of rbeans

- Competition 2006: Jambox (2 $\times$ self-labeling, matrix interpret.)
- Competition 2007: no proof

Example (SRS/Waldmann/rbeans)

$$\mathcal{R}_{rbeans} = \{baa \to abc,\ ca \to ac,\ cb \to ba,\ \epsilon \to= b\}$$

$$[a] = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{5}{2} & 6 \\ 0 & 0 & 1 \end{pmatrix},\ [b] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ [c] = \begin{pmatrix} 1 & 2 & 0 \\ 0 & \frac{5}{2} & \frac{7}{2} \\ 0 & 0 & 1 \end{pmatrix},\ [\epsilon] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[\mathcal{R}] = \left\{ \begin{pmatrix} 0 & \frac{1}{4} & \frac{17}{4} \\ 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{3}{2} & \frac{17}{2} \\ 0 & 0 & \frac{15}{4} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

removable rules: $baa \to abc$ and $ca \to ac$

remains: $\mathcal{R}'_{rbeans} = \{cb \to ba,\ \epsilon \to= b\}$

Proof of SRS/Waldmann07b/size-12-alpha-3-num-552

- not used in competition
- no participant of 2006 competition able to prove termination

Example (SRS/Waldmann07b/size-12-alpha-3-num-552)

$\mathcal{R}_{num-552} = \{ab \to \epsilon,\ ac \to bccaab,\ bc \to \epsilon\}$

$$[a] = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ [b] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ [c] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[\mathcal{R}] = \left\{ \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

removable rules: $ac \to bccaab$

remains: $\mathcal{R}'_{num-552} = \{ab \to \epsilon,\ bc \to \epsilon\}$

Implementation: Evolutionary Programming (EP)

- initial genotype: interpretation starting with

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & & \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

- phenotype: interpretation of rules
- whithout recombination operator
- two mutation operators
  - M_1^ξ : reset random entry
  - M_2^ξ : set random value $x' := \xi([\min(0, x - \Delta), \ x + \Delta])$
- fitness: sum up penalties for unwanted properties
  - negative entries (e^2)
  - all rule interpretations are 0 (c)
- valid interpretation iff sum is zero

Implementation: SAT Solving

- interpret every unknown as a **binary fixed point** variable $x = m \cdot 2^e$, e.g. $e = -1$ (fixed)
- calculate $[l] - [r]$ for each rule
  (Note: Need higher precision for products.)
- transform $[l] - [r]$ to boolean satisfiability problem
- use **Minisat**

Open Questions/Discussion

Open Questions

- compare power of matrix interpretations over $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{A}lg$, $\mathbb{R}$
- compare power of matrix interpretations for fixed dimensions
- integer matrix proof for $z057$ with dimension 4?

Related Work: Salvador Lucas 'On the relative power of polynomials with real, rational and integer coefficients in proofs of termination of rewriting' (2006)