Termination of String Rewriting with Matrix Interpretations

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Ring Interpretations

homomorphism: $\Sigma^* \rightarrow$ ordered ring $(D, +, \cdot, >)$

- multiplication (\Rightarrow concatenation)
- addition/subtraction (\Rightarrow rule application)
- For step $xly \rightarrow_R xry$, $i(xly) - i(xry) = i(x) \cdot (i(l) - i(r)) \cdot i(y)$. Proof obligation for termination of system *R*:

$$i(\Sigma^*) \cdot i(R) \cdot i(\Sigma^*) \in P$$

 $\dots \text{ where } (D, >) \text{ well-founded, } P := \{ x \mid x > 0 \}, \\ i(R) := \{ i(l) - i(r) \mid (l \to r) \in R \} \text{ RTA, Seattle, August 2006 - p.3/2}$

Admissable Differences

Given a set $A = i(\Sigma)$ of positive ring elements, define $\operatorname{core}(A) := \{d \mid A^* \cdot d \cdot A^* \subseteq P\}.$

Definition: $i : \Sigma^* \to D$ is an A-interpretation for Riff $i(\Sigma) \subseteq A$ and $i(R) \subseteq \operatorname{core}(A)$.

Theorem: For rewriting systems R and S over Σ : if i is A-interpretation for R with $i(S) \subseteq P \cup \{0\}$, then R terminates relative to S.

Illustration of Theorem

use $M := \{m \mid \forall i \exists j : m_{i,j} > 0\}$. (in each row, at least one positive entry). Then core(M) = M.

$$R = \{aa \to aba\}, S = \{b \to bb\}.$$

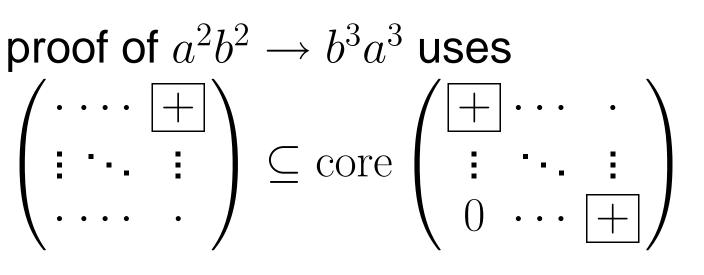
$$M\text{-interpretation } i: a \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$
$$i(aa \to aba) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$
$$i(b \to bb) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

R terminates relative to S.

More interesting matrix shapes

$$E := \{ m \mid \forall i : m_{i,i} > 0 \}. \text{ Then } \operatorname{core}(E) = P.$$
$$\binom{* + }{* + } \subseteq \operatorname{core} \binom{+ }{* + }$$

also works for subsets of indices: if $B = \begin{pmatrix} A \\ * \end{pmatrix}$, then $\begin{pmatrix} \operatorname{core}(A) \\ * \end{pmatrix} \subseteq \operatorname{core}(B)$.



Performance of Matrix Tools

(percentage of YES in 2006 SRS competition)

- MultumNonMulta (Dieter Hofbauer): uses only the matrix method: 51 %
- Matchbox/Satelite (J. W.): labelling, matrices, RFC match-bounds: 68 %
- Torpa (Hans Zantema): various techniques, including 3×3 matrices: 75 %
- Jambox (Jörg Endrullis): \approx Matchbox + dependency pairs: 94 %

Implementation (1)

random guesses or complete enumeration:

Torpa uses matrix shape $\begin{pmatrix} 0 & * + \\ 0 & * & * \\ 0 & 0 & 0 \end{pmatrix} \subseteq \operatorname{core} \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}$

with $* \in \{0, 1, 4\}$, in 36% of its proofs, e.g. z007: TORPA 1.6 is applied to

ab -> ba, ba -> aacb,

[A] Choose interpretation in NxN,

order : (x,y) > (x',y') <=> x > x' & y >= y'

- a : lambda (x,y) . (x+y,4y)
- b : lambda (x,y) . (x,4y+1)
- c : lambda (x,y) . (x,0)

remove: a b -> b a

Implementation (2)

MultumNonMulta: GNU Linear Programming Kit for shape $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and "sophisticated guesses" for larger dimensions, e. g. r10: $\{ba^2b \to a^4, ab^2a \to b^4\}/\{b \to b^3\}$ 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 b: 1000000000000000 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 00 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 \cap 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0

(found in 250 seconds)

Implementation (3)

- fix dimension, say 5. \Rightarrow constraint system with $|\Sigma| \cdot d^2$ unknowns (entries in interpretation) and $|R| \cdot d^2$ constraints (entries in differences)
- fix maximal value for entries, say 7. \Rightarrow finite domain constraint system
- represent unknowns in binary ⇒ boolean satisfiability problem, (15.000 variables, 90.000 clauses, 300.000 literals) ⇒ solve by SAT solver (SateliteGTI) (z001 takes 7 seconds)
- Jambox: linear programming + SAT solving, Matchbox: likewise, but ...

Implementation (4)

... Matchbox uses only one bit per matrix entry (computation in $\{0,1\} \subset \mathbb{N}$, so 1+1 is "forbidden")

Limits (I): Growth

- Entries in powers of a fixed matrix are bounded by an exponential function of the exponent.
- There can be no strictly monotone matrix interpretation for a rewriting system with longer than exponential derivations.
- But there could be matrix proof by step-wise removal of rules.
- It cannot, for systems with long derivations where each rule occurs (roughly) equally often,

e.g.
$$\{ab \rightarrow bca, cb \rightarrow bbc\}$$
 (z018, z020)

Limits (II): Dimension

Matrix rings obey certain polynomial identities, e.g.

- dim 1, [A, B] = 0. No one-dimensional termination proof for $\{ab \rightarrow ba\}$
- dim 2, $[[A, B]^2, C] = 0$. No 2-dim proof for

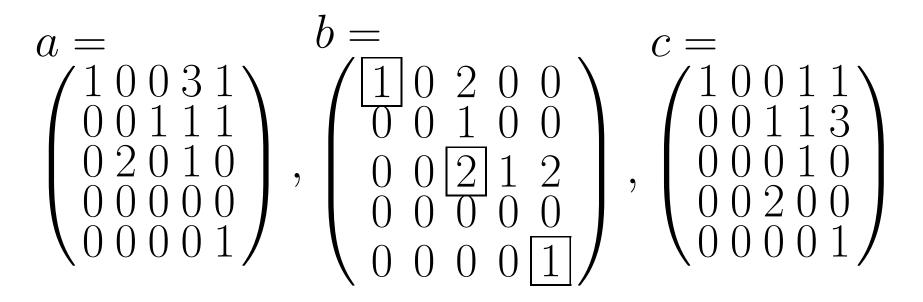
 $\{abcbc \rightarrow cbcba, acbcb \rightarrow bcbca, bccba \rightarrow abccb, cbbca \rightarrow acbbc\}.$

(Notation uses commutator [A, B] := AB - BA) Consider SRS hierarchy defined by "minimal matrix proof dimension":

- is every level inhabited?
- what levels are decidable?

Zantema's "other" Problem

 ${a^2 \rightarrow bc, b^2 \rightarrow ac, c^2 \rightarrow ab} = RTALOOP 104 = z086$, solved by strictly monotone interpretation



This interpretation grows exponentially (see). Exact complexity of z086 is open. suggest RTALOOP 104': is it polynomial?