Matrix Interpretations for Proving Termination of Term Rewriting

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Motivation

Matrix interpretations for termination proofs for string rewriting were developed by Hofbauer and Waldmann [HW06]. It allowed them to prove termination for $\{aa \to bc, bb \to ac, cc \to ab\}$.

We extend this approach to:

- term rewriting
  by taking linear combinations of matrix interpretations
- dependency pairs [GTSK05, HM04] and top-termination

Outline

1. Preliminaries
   - Term rewriting systems
   - (Relative) termination
2. Monotone algebras
   - Σ-algebra and monotonicity
   - Weakly monotone algebras
3. Matrix interpretations
   - Relative termination
   - Top-termination and Dependency Pairs
   - Implementation and Performance Measurements

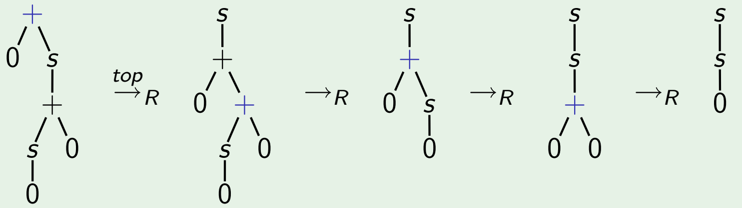
Term rewriting systems (TRS)

Let $\mathcal{X}$ be a set of variable symbols, we use $x, y, z, \ldots$ as variables.

- A signature Σ is a ranked alphabet with arity ar: $\Sigma \to \mathbf{N}$.
- The set of terms $\mathcal{T}(\Sigma, \mathcal{X}) \approx$ finite trees over Σ, $\mathcal{X}$.
- A TRS R is a set of rewrite rules $\ell \to r \in \mathcal{T}(\Sigma, \mathcal{X}) \times \mathcal{T}(\Sigma, \mathcal{X})$.

Example ($\Sigma = \{0, \mathrm{s}, +\}$ with $\mathrm{ar}(0) = 0,\ \mathrm{ar}(s) = 1,\ \mathrm{ar}(+) = 2$)

$$R = \{\ +(x, \mathrm{s}(y)) \to \mathrm{s}(+(x, y)),$$
$$+(x, 0) \to x\ \}$$



Definition (Termination, SN($\to$))

A relation $\to$ is well-founded, terminating or strong normalization (SN($\to$)) if no infinite sequence $t_1 \to t_2 \to t_3 \to \ldots$ exists.

Definition (Relative termination, SN($\to_1$ / $\to_2$))

A binary relation $\to_1$ is terminating relative to a binary relation $\to_2$ if no $\to_1 \cup \to_2$ sequence contains infinitely many $\to_1$ steps.

Observe that SN(R) $\Leftrightarrow$ SN($R/\emptyset$).

Definition (Σ-algebra $(A, [\cdot])$)

A Σ-algebra $(A, [\cdot])$ consists of a set A,
and for every $f \in \Sigma$ an interpretation function $[f]\colon A^{\mathrm{ar}(f)} \to A$.

Example (Proving $\mathrm{SN}(R)$... usual approach: polynomials over $\mathbf{N}$)

$$R = \{ \quad +(x, \mathrm{s}(y)) \to \mathrm{s}(+(x, y)), \quad +(x, 0) \to x \quad \}$$

Let $A = \mathbf{N}$ and

$$[0] = 1 \qquad [\mathrm{s}](x) = x + 1 \qquad [+](x, y) = x + 2 \cdot y$$

Terms are functions depending on the values of their variables:

$$[+(x, \mathrm{s}(y)), \alpha] = \alpha(x) + 2 \cdot \alpha(y) + 2 \quad [+(x, 0), \alpha] = \alpha(x) + 2$$
$$[\mathrm{s}(+(x, y)), \alpha] = \alpha(x) + 2 \cdot \alpha(y) + 1 \quad [x, \alpha] = \alpha(x)$$

Both rules are strictly decreasing w.r.t. $>$ for all $\alpha\colon \mathcal{X} \to \mathbf{N}$ and $[0]$, $[\mathrm{s}]$, $[+]$ are monotone in all arguments. This implies $\mathrm{SN}(R)$.

Example (SN(R/S))

Consider $R = \{f(f(x)) \to f(g(f(x)))\}$, $S = \{f(x) \to g(f(x))\}$.

We choose $A = \mathbf{N}^2$, symbol interpretations:

$$[\mathrm{f}](\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad [\mathrm{g}](\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $\alpha \colon \mathcal{X} \to A$ be arbitrary; write $\vec{x} = \alpha(x)$. We obtain

$$[f(f(x))] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} >^? \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = [f(g(f(x)))]$$

$$[f(x)] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} >^? \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = [g(f(x))]$$

Both rules seem to be decreasing? But S does not terminate!

We need two relations $>$ (strictly dec.), $\gtrsim$ (weakly dec.)
where $\gtrsim$ is not the union of $>$ and equality.

Weakly monotone Σ-algebras

Definition (Weakly monotone Σ-algebra $(A, [\cdot], >, \gtrsim)$)

. . . is a Σ-algebra $(A, [\cdot])$ equipped with relations $>$, $\gtrsim$ on A s.t.

- $>$ is well-founded,
- $> \cdot \gtrsim \subseteq >$ (compatibility), and
- for all $f \in \Sigma$ the function $[f]$ is monotone with respect to $\gtrsim$

Definition (Extended monotone Σ-algebra $(A, [\cdot], >, \gtrsim)$)

. . . is a weakly monotone Σ-algebra in which moreover

- for all $f \in \Sigma$ the function $[f]$ is monotone with respect to $>$

A function $[f]$ is monotone w.r.t. $\gtrsim$ on A iff $a_j \gtrsim b_j$ always implies

$$[f](a_1, \dots, a_j, \dots, a_n) \gtrsim [f](a_1, \dots, b_j, \dots, a_n)$$

Let $(A, [\cdot])$ be a Σ-algebra and $>$ a binary relation on A.
We extend $>$ from A to $\mathcal{T}(\Sigma, \mathcal{X})$ by

$$\ell > r \iff [\ell, \alpha] > [r, \alpha] \ \ \forall \alpha \colon \mathcal{X} \to A$$

Let R, S be TRSs over a signature Σ.

Theorem (Relative termination)

Let $(A, [\cdot], >, \gtrsim)$ be an extended monotone Σ-algebra such that

- $R \subseteq \gtrsim$ *and* $S \subseteq \gtrsim$,

then $\mathsf{SN}((R \setminus >)/(S \setminus >))$ *implies* $\mathsf{SN}(R/S)$.

Theorem (Top-termination)

Let $(A, [\cdot], >, \gtrsim)$ be a weakly monotone Σ-algebra such that

- $R \subseteq \gtrsim$ *and* $S \subseteq \gtrsim$,

then $\mathsf{SN}((R \setminus >)_{top}/S)$ *implies* $\mathsf{SN}(R_{top}/S)$.

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Matrix interpretations for (relative) termination

We construct extended monotone algebras $(\mathbf{N}^d, [\cdot], >, \gtrsim)$.
We define $>$ and $\gtrsim$ on $\mathbf{N}^d$ by

$$(v_1, \ldots, v_d) > (u_1, \ldots, u_d) \iff v_1 > u_1 \ \wedge \ \forall i : v_i \geq u_i$$
$$(v_1, \ldots, v_d) \gtrsim (u_1, \ldots, u_d) \iff \forall i : v_i \geq u_i$$

As interpretations $[f]$ we choose

$$[f](\vec{v_1}, \ldots, \vec{v_n}) \ = \ F_1\vec{v_1} + \cdots + F_n\vec{v_n} + \vec{f}$$

- matrices $F_1, \ldots, F_n \in \mathbf{N}^{d \times d}$ with $\forall i$: $(F_i)_{1,1} \geq 1$, and
- a vector $\vec{f} \in \mathbf{N}^d$

Note that $\gtrsim$ does not coincide with the union of $>$ and equality.

Example

Consider $R = \{f(f(x)) \to f(g(f(x)))\}$, $S = \{f(x) \to g(f(x))\}$.

We choose $A = \mathbf{N}^2$, symbol interpretations:

$$[\mathrm{f}](\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad [\mathrm{g}](\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $\alpha \colon \mathcal{X} \to A$ be arbitrary; write $\vec{x} = \alpha(x)$. We obtain

$$[f(f(x))] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} > \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = [f(g(f(x)))]$$

$$[f(x)] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \gtrsim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = [g(f(x))]$$

We conclude $\mathrm{SN}(R/S)$.

Observe that there exists no extended monotone algebra in which $\gtrsim$ coincides with the union of $>$ and equality. Hence for relative termination the general notion of extended monotone algebra is essential; well-founded monotone algebras are not sufficient.

Dependency Pairs

Let R be a TRS over Σ. We define

- the defined symbols $\mathrm{D}(R) = \{\mathrm{root}(\ell) \mid \ell \to r \in R\}$,

For every $f \in \Sigma$ let $f_\#$ be a fresh symbol with the same arity as f.
By $t_\#$ we denote $f_\#(t_1, \ldots, t_n)$ for $t = f(t_1, \ldots, t_n) \in \mathcal{T}(\Sigma, \mathcal{X})$.

Definition (Dependency Pairs [GTSK05, HM04])

The set of dependency pairs of R is defined by

$$\mathrm{DP}(R) = \{\ell_\# \to r'_\# \mid \ell \to r \in R,\ r' \trianglelefteq r \text{ with } \mathrm{root}(r) \in D(R)\}$$

Theorem

We have $\mathrm{SN}(R) \Leftrightarrow \mathrm{SN}(\mathrm{DP}(R)_{top}/R)$.

Matrix interpretations for top-termination

We construct weakly monotone algebras $(\mathbf{N}^d \cup \mathbf{N}, [\cdot], >, \gtrsim)$.

On $\mathbf{N}^d$ we define $>= \emptyset$,

$$(v_1, \ldots, v_d) \gtrsim (u_1, \ldots, u_d) \iff \forall i : v_i \geq u_i$$

and on $\mathbf{N}$ we define $> = >_{\mathbf{N}}$, $\gtrsim = \geq_{\mathbf{N}}$.

For $f \in \Sigma$ we choose $F_1, \ldots, F_n \in \mathbf{N}^{d \times d}$ and $\vec{f} \in \mathbf{N}^d$:

$$[f](\vec{v_1}, \ldots, \vec{v_n}) = F_1 \vec{v_1} + \cdots + F_n \vec{v_n} + \vec{f}$$

For $f_\# \in \Sigma_\#$ we choose row vectors $\vec{f_1}, \ldots, \vec{f_1} \in \mathbf{N}^d$ and $c_f \in \mathbf{N}$

$$[f_\#](\vec{v_1}, \ldots, \vec{v_n}) = \vec{f_1} \vec{v_1} + \cdots + \vec{f_n} \vec{v_n} + c_f$$

Example

Consider the TRS consisting of the following rule.

$$\mathrm{g}(\mathrm{g}(\mathrm{s}(x), y), \mathrm{g}(z, u)) \to \mathrm{g}(\mathrm{g}(y, z), \mathrm{g}(x, \mathrm{s}(u)))$$

We have 3 dependency pairs:

1. $\mathrm{g}_{\#}(\mathrm{g}(\mathrm{s}(x), y), \mathrm{g}(z, u)) \to \mathrm{g}_{\#}(\mathrm{g}(y, z), \mathrm{g}(x, \mathrm{s}(u)))$
2. $\mathrm{g}_{\#}(\mathrm{g}(\mathrm{s}(x), y), \mathrm{g}(z, u)) \to \mathrm{g}_{\#}(y, z)$
3. $\mathrm{g}_{\#}(\mathrm{g}(\mathrm{s}(x), y), \mathrm{g}(z, u)) \to \mathrm{g}_{\#}(x, \mathrm{s}(u))$

(2) and (3) can easily be removed by counting the symbols, i.e.

- $[\mathrm{g}_{\#}](x, y) = [\mathrm{g}](x, y) = 1 + x + y$
- $[\mathrm{s}](x) = x + 1$

as polynomial interpretation over $\mathbf{N}$.

The original rule and the first dependency pair remain...

We choose dimension $d = 2$ ($A_s = \mathbf{N}^2$, $A_\# = \mathbf{N}$) with

$$[\mathrm{g}_\#](\vec{x_0}, \vec{x_1}) = (1,0) \cdot \vec{x_0} + (0,1) \cdot \vec{x_1}$$

$$[\mathrm{g}](\vec{x_0}, \vec{x_1}) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \vec{x_0} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{x_1}$$

$$[\mathrm{s}](\vec{x_0}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \vec{x_0} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- $\mathrm{g}(\mathrm{g}(\mathrm{s}(x), y), \mathrm{g}(z, u)) \to \mathrm{g}(\mathrm{g}(y, z), \mathrm{g}(x, \mathrm{s}(u)))$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \vec{y} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \vec{z} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{u} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\gtrsim$$
$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \vec{y} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \vec{z} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \vec{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- $\mathrm{g}_\#(\mathrm{g}(\mathrm{s}(x), y), \mathrm{g}(z, u)) \to \mathrm{g}_\#(\mathrm{g}(y, z), \mathrm{g}(x, \mathrm{s}(u)))$

$$(1,0) \cdot \vec{x} + (1,0) \cdot \vec{y} + (1,0) \cdot \vec{z} + (0,1) \cdot \vec{u} + (1)$$
$$>$$
$$(1,0) \cdot \vec{x} + (1,0) \cdot \vec{y} + (1,0) \cdot \vec{z} + (0,0) \cdot \vec{u} + (0)$$

Example (Variant of TPDB 2006/secret2006/jambox - 5)

We want to prove SN(R) for

$$R = \{\ \mathrm{a}(\mathrm{a}(y,0),0) \to y,\quad \mathrm{c}(\mathrm{a}(\mathrm{c}(\mathrm{c}(y)),x)) \to \mathrm{a}(\mathrm{c}(\mathrm{c}(\mathrm{c}(\mathrm{a}(x,0)))),y)\ \}$$

There is only one interesting dependency pair:

$$\mathrm{c}_{\#}(\mathrm{a}(\mathrm{c}(\mathrm{c}(y)),x)) \to \mathrm{c}_{\#}(\mathrm{c}(\mathrm{c}(\mathrm{a}(x,0))))$$

The following symbol interpretations prove termination:

$$[\mathrm{a}](\vec{x_0},\vec{x_1}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{x_0} + \begin{pmatrix} 2 & 3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \cdot \vec{x_1} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[\mathrm{g}](\vec{x}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad [0] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[\mathrm{c}_{\#}](\vec{x}) = (1,0,3) \cdot \vec{x} + (1)$$

Implementation

The basic algorithm finds a matrix interpretation that allows to remove rules from a termination problem. The inputs are:

- a pair of rewrite systems (R, S) over signature Σ
- a goal $g \in \{\mathrm{Full}, \mathrm{Top}\}$ for $\mathsf{SN}(R/S)$ or $\mathsf{SN}(R_{\mathsf{top}}/S)$
- dimension d, initial bits b, result bits b'

The implementation of the algorithm has two stages:

1. It produces a system I of inequalities between polynomials of unknowns (constraints on coefficients in vectors and matrices)
2. By putting bounds $(2^b - 1)$ on the range of the variables, the problem becomes finite and can be translated into a boolean satisfiability problem F. Then we call a SAT solver (SatELiteGTI, [EB05]) to find a satisfying assignment.

Performance of the matrix method on the TPDB 2005

method	dimension d	initial bits b	result bits b'	YES
direct	1	4	5	141
direct	2	2	3	219
direct	3	3	4	225
dependency pairs	1	4	5	433
dependency pairs	2	1	2	503
dependency pairs	2	2	3	505
dependency pairs	3	2	3	507
dependency pairs	4	2	3	509
dependency pairs +	4	2	3	538

- direct method = pure matrix interpretations
- dependency pairs = combination of matrix interpretations with the dependency pairs framework (DP graph approximation, usable rules criterion [GTSK05, HM04] and the sub-term criterion [HM04])
- dependency pairs + = extension by the transformation of applicative TRSs into functional form [GTSK05], and rewriting of right hand sides [Zan05]

Conclusions

The idea of matrix interpretations for termination proofs for string rewriting was developed by Hofbauer and Waldmann [HW06]. It allowed them to prove termination for $\{aa \to bc, bb \to ac, cc \to ab\}$.

In this paper we showed how to extend this approach to term rewriting successfully. A crucial ingredient is taking linear combinations of matrix interpretations for symbols of arity > 1.

In the results on the benchmark database TPDB we see a big jump when increasing the dimension from 1 (representing linear polynomial interpretations) to 2. Increasing the dimension from 2 to higher values only yields a minor improvement, while then the sizes of the satisfiability formulas strongly increase.

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